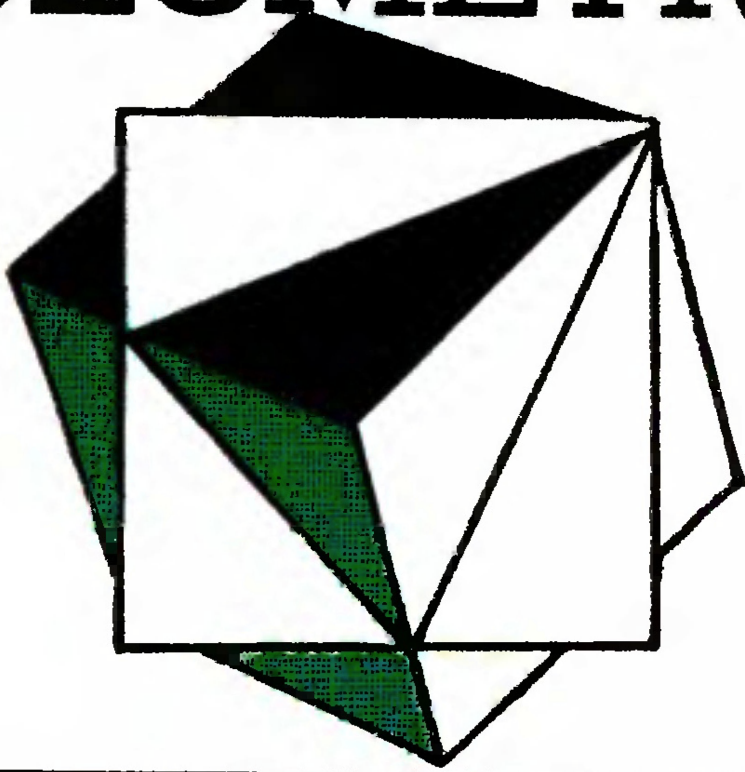
SCIENCE FOR EVERYONE

I.F. SHARYGIN

PROBLEMS IN SOLID GEOMETRY



MIR

Science
for Everyone

И.Ф. Шарыгин

Задачи по геометрии
Стереометрия

Издательство «Наука», Москва

I.F. Sharygin

Problems in Solid Geometry

Translated from the Russian
by Leonid Levant

Mir
Publishers
Moscow

First published 1986
Revised from the 1984 Russian edition

На английском языке

Contents

Preface

This book contains 340 problems in solid geometry and is a natural continuation of *Problems in Plane Geometry*, Nauka, Moscow, 1982. It is therefore possible to confine myself here to those points where this book differs from the first.

The problems in this collection are grouped into (1) computational problems and (2) problems on proof.

The simplest problems in Section 1 only have answers, others, have brief hints, and the most difficult, have detailed hints and worked solutions. There are two reservations. Firstly, in most cases only the general outline of the solution is given, a number of details being suggested for the reader to consider. Secondly, although the suggested solutions are valid, they are not patterns (models) to be used in examinations.

Sections 2-4 contain various geometric facts and theorems, problems on maximum and minimum (some of the problems in this part could have been put in Section 1), and problems on loci. Some questions pertaining to the geometry of tetrahedron, spherical geometry, and so forth are also considered here.

As to the techniques for solving all these problems, I have to state that I prefer analytical computational methods to those associated with plane geometry. Some of the difficult problems in solid geometry will require a high level of concentration from the reader, and an ability to carry out some rather complicated work.

The Author

Section 1

Computational Problems

1. Given a cube with edge a. Two vertices of a regular tetrahedron lie on its diagonal and the two remaining vertices on the diagonal of its face. Find the volume of the tetrahedron.

2. The base of a quadrangular pyramid is a rectangle, the altitude of the pyramid is h. Find the volume of the pyramid if it is known that all five of its faces are equivalent.

3. Among pyramids having all equal edges (each of length a), find the volume of the one which has the greatest number of edges.

4. Circumscribed about a ball is a frustum of a regular quadrangular pyramid whose slant height is equal to a. Find its lateral surface area.

5. Determine the vertex angle of an axial section of a cone if its volume is three times the volume of the ball inscribed in it.

6. Three balls touch the plane of a given triangle at the vertices of the triangle and one another. Find the radii of these balls if the sides of the triangle are equal to a, b, and c.

7. Find the distance between the skew diagonals of two neighbouring faces of a cube with edge a. In what ratio is each of these diagonals divided by their common perpendicular?

8. Prove that the area of the projection of a polygon situated in the plane α on the plane β

is equal to $S \cos \varphi$, where S denotes the plane of the polygon and φ the angle between the planes α and β.

9. Given three straight lines passing through one point A. Let B_1 and B_2 be two points on one line, C_1 and C_2 two points on the other, and D_1 and D_2 two points on the third line. Prove that

$$\frac{V_{AB_1C_1D_1}}{V_{AB_2C_2D_2}} = \frac{|AB_1| \cdot |AC_1| \cdot |AD_1|}{|AB_2| \cdot |AC_2| \cdot |AD_2|}.$$

10. Let α, β, and γ denote the angles formed by an arbitrary straight line with three pairwise perpendicular lines. Prove that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

11. Let S and P denote the areas of two faces of a tetrahedron, a the length of their common edge, and α the dihedral angle between them. Prove that the volume V of the tetrahedron can be found by the formula

$$V = \frac{2SP \sin \alpha}{3a}.$$

12. Prove that for the volume V of an arbitrary tetrahedron the following formula is valid: $V = \frac{1}{6} abd \sin \varphi$, where a and b are two opposite edges of the tetrahedron, d the distance between them, and φ the angle between them.

13. Prove that the plane bisecting the dihedral angle at a certain edge of a tetrahedron divides the opposite edge into parts proportional to the areas of the faces enclosing this angle.

14. Prove that for the volume V of the polyhedron circumscribed about a sphere of radius R

the following equality holds: $V = \frac{1}{3}S_n R$, where S_n is the total surface area of the polyhedron.

15. Given a convex polyhedron all of whose vertices lie in two parallel planes. Prove that its volume can be computed by the formula

$$V = \frac{h}{6}(S_1 + S_2 + 4S),$$

where S_1 is the area of the face situated in one plane. S_2 the area of the face situated in the other plane, S the area of the section of the polyhedron by the plane equidistant from the two given planes, and h is the distance between the given planes.

16. Prove that the ratio of the volumes of a sphere and a frustum of a cone circumscribed about it is equal to the ratio of their total surface areas.

17. Prove that the area of the portion of the surface of a sphere enclosed between two parallel planes cutting the sphere can be found by the formula

$$S = 2\pi Rh,$$

where R is the radius of the sphere and h the distance between the planes.

18. Prove that the volume of the solid generated by revolving a circular segment about a nonintersecting diameter can be computed by the formula

$$V = \frac{1}{6}\pi a^2 h,$$

where a is the length of the chord of this segment and h the projection of this chord on the diameter.

19. Prove that the line segments connecting the vertices of a tetrahedron with the median points of opposite faces intersect at one point (called the centre of gravity of the tetrahedron) and are divided by this point in the ratio 3 : 1 (reckoning from the vertices).

Prove also that the line segments joining the midpoints of opposite edges intersect at the same point and are bisected by this point.

20. Prove that the straight lines joining the midpoint of the altitude of a regular tetrahedron to the vertices of the face onto which this altitude is dropped are pairwise perpendicular.

21. Prove that the sum of the squared lengths of the edges of a tetrahedron is four times the sum of the squared distances between the midpoints of its skew edges.

22. Given a cube $ABCDA_1B_1C_1D_1$* with an edge a, in which K is the midpoint of the edge DD_1. Find the angle and the distance between the straight lines CK and A_1D.

23. Find the angle and the distance between two skew medians of two lateral faces of a regular tetrahedron with edge a.

24. The base of the pyramid $SABCD$ is a quadrilateral $ABCD$. The edge SD is the altitude of the pyramid. Find the volume of the pyramid if it is known that $|AB| = |BC| = \sqrt{5}$, $|AD| =$

---

* $ABCD$ and $A_1B_1C_1D_1$ are two faces of the cube, AA_1, BB_1, CC_1, DD_1 are its edges.

$|DC| = \sqrt{2}$, $|AC| = 2$, $|SA| + |SB| = 2 + \sqrt{5}$.

25. The base of a pyramid is a regular triangle with side a, the lateral edges are of length b. Find the radius of the ball which touches all the edges of the pyramid or their extensions.

26. A sphere passes through the vertices of one of the faces of a cube and touches the sides of the opposite faces of the cube. Find the ratio of the volumes of the ball and the cube.

27. The edge of the cube $ABCDA_1B_1C_1D_1$ is equal to a. Find the radius of the sphere passing through the midpoints of the edges AA_1, BB_1, and through the vertices A and C_1.

28. The base of a rectangular parallelepiped is a square with side a, the altitude of the parallelepiped is equal to b. Find the radius of the sphere passing through the end points of the side AB of the base and touching the faces of the parallelepiped parallel to AB.

29. A regular triangular prism with a side of the base a is inscribed in a sphere of radius R. Find the area of the section of the prism by the plane passing through the centre of the sphere and the side of the base of the prism.

30. Two balls of one radius and two balls of another radius are arranged so that each ball touches three other balls and a given plane. Find the ratio of the radii of the greater and smaller balls.

31. Given a regular tetrahedron $ABCD$ with edge a. Find the radius of the sphere passing through the vertices C and D and the midpoints of the edges AB and AC.

32. One face of a cube lies in the plane of the base of a regular triangular pyramid. Two vertices of the cube lie on one of the lateral faces of the pyramid and another two on the other two faces (one vertex per face). Find the edge of the cube if the side of the base of the pyramid is equal to a and the altitude of the pyramid is h.

33. The dihedral angle at the base of a regular n-gonal pyramid is equal to α. Find the dihedral angle between two neighbouring lateral faces.

34. Two planes are passed in a triangular prism $ABCA_1B_1C_1$*: one passes through the vertices A, B, and C_1, the other through the vertices A_1, B_1, and C. These planes separate the prism into four parts. The volume of the smallest part is equal to V. Find the volume of the prism.

35. Through the point situated at a distance a from the centre of a ball of radius R ($R > a$), three pairwise perpendicular chords are drawn. Find the sum of the squared lengths of the segments of the chords into which they are divided by the given point.

36. The base of a regular triangular prism is a triangle ABC with side a. Taken on the lateral edges are points A_1, B_1, and C_1 situated at distances $a/2$, a, and $3a/2$, respectively, from the plane of the base. Find the angle between the planes ABC and $A_1B_1C_1$.

37. The side of the base of a regular quadrangular pyramid is equal to the slant height of a lateral face. Through a side of the base a cutting plane is passed separating the surface of the pyra-

* Here and henceforward, ABC and $A_1B_1C_1$ are the bases of the prism and AA_1, BB_1, CC_1 its lateral edges.

mid into two equal portions. Find the angle between the cutting plane and the plane of the base of the pyramid.

38. The centre of a ball is found in the plane of the base of a regular triangular pyramid. The vertices of the base lie on the surface of the ball. Find the length l of the line of intersection of the surfaces of the ball and pyramid if the radius of the ball is equal to R, and the plane angle at the vertex of the pyramid is equal to α.

39. In a regular hexagonal pyramid $SABCDEF$ (S the vertex), on the diagonal AD, three points are taken which divide the diagonal into four equal parts. Through these division points sections are passed parallel to the plane SAB. Find the ratios of the areas of the obtained sections.

40. In a regular quadrangular pyramid, the plane angle at the vertex is equal to the angle between the lateral edges and the plane of the base. Determine the dihedral angles between the adjacent lateral faces of this pyramid.

41. The base of a triangular pyramid all of whose lateral edges are pairwise perpendicular is a triangle having an area S. The area of one of the lateral faces is Q. Find the area of the projection of this face on the base.

42. $ABCA_1B_1C_1$ is a regular triangular prism all of whose edges are equal to one another. K is a point on the edge AB different from A and B, M is a point on the straight line B_1C_1, and L is a point in the plane of the face ACC_1A_1. The straight line KL makes equal angles with the planes ABC and ABB_1A_1, the line LM makes equal angles with the planes BCC_1B_1 and ACC_1A_1, the line KM also makes equal angles with the

planes BCC_1B_1 and ACC_1A_1. It is known that $|KL| = |KM| = 1$. Find the edge of the prism.

43. In a regular quadrangular pyramid, the angle between the lateral edges and the plane of the base is equal to the angle between a lateral edge and a plane of the lateral face not containing this edge. Find this angle.

44. Find the dihedral angle between the base and a lateral face of a frustum of a regular triangular pyramid if it is known that a ball can be inscribed in it, and, besides, there is a ball which touches all of its edges.

45. Each of three edges of a triangular pyramid is equal to 1, and each of three other edges is equal to a. None of the faces is a regular triangle. What is the range of variation of a? What is the volume of this pyramid?

46. The lateral faces of a triangular pyramid are equivalent and are inclined to the plane of the base at angles α, β, and γ. Find the ratio of the radius of the ball inscribed in this pyramid to the radius of the ball touching the base of the pyramid and the extensions of the three lateral faces.

47. All edges of a regular hexagonal prism are equal to a (each). Find the area of the section passed through a side of the base at an angle α to the plane of the base.

48. In a rectangular parallelepiped $ABCDA_1B_1C_1D_1$, $|AB| = a$, $|AD| = b$, $|AA_1| = c$. Find the angle between the planes AB_1D_1 and A_1C_1D.

49. The base of the pyramid $ABCDM$ is a square with base a, the lateral edges AM and BM are also equal to a (each). The lateral edges CM

and DM are of length b. On the face CDM as on the base a triangular pyramid $CDMN$ is constructed outwards, each lateral edge of which has a length a. Find the distance between the straight lines AD and MN.

50. In a tetrahedron, one edge is equal to a, the opposite edge to b, and the rest of the edges to c. Find the radius of the circumscribed ball.

51. The base of a triangular pyramid is a triangle with sides a, b, and c; the opposite lateral edges of the pyramid are respectively equal to m, n, and p. Find the distance from the vertex of the pyramid to the centre of gravity of the base.

52. Given a cube $ABCDA_1B_1C_1D_1$; through the edge AA_1 a plane is passed forming equal angles with the straight lines BC_1 and B_1D. Find these angles.

53. The lateral edges of a triangular pyramid are pairwise perpendicular, one of them being the sum of two others is equal to a. Find the radius of the ball touching the base of the pyramid and the extensions of its lateral faces.

54. The base of a triangular pyramid $SABC$ is a regular triangle ABC with side a, the edge SA is equal to b. Find the volume of the pyramid if it is known that the lateral faces of the pyramid are equivalent.

55. The base of a triangular pyramid $SABC$ is an isosceles triangle ABC ($\hat{A} = 90°$). The angles $\widehat{SAB}$, $\widehat{SCA}$, $\widehat{SAC}$, $\widehat{SBA}$ (in the indicated order) form an arithmetic progression whose difference is not equal to zero. The areas of the faces

SAB, ABC and SAC form a geometric progression. Find the angles forming an arithmetic progression.

56. The base of a triangular pyramid $SABC$ is a regular triangle ABC with side a. Find the volume of this pyramid if it is known that $\widehat{ASC} = \widehat{ASB} = \alpha$, $\widehat{SAB} = \beta$.

57. In the cube $ABCDA_1B_1C_1D_1$ K is the midpoint of the edge AA_1, the point L lies on the edge BC. The line segment KL touches the ball inscribed in the cube. In what ratio is the line segment KL divided by the point of tangency?

58. Given a tetrahedron $ABCD$ in which $\widehat{ABC} = \widehat{BAD} = 90°$. $|AB| = a$, $|DC| = b$, the angle between the edges AD and BC is equal to α. Find the radius of the circumscribed ball.

59. An edge of a cube and an edge of a regular tetrahedron lie on the same straight line, the midpoints of the opposite edges of the cube and tetrahedron coincide. Find the volume of the common part of the cube and tetrahedron if the edge of the cube is equal to a.

60. In what ratio is the volume of a triangular pyramid divided by the plane parallel to its two skew edges and dividing one of the other edges in the ratio $2:1$?

61. In a frustum of a regular quadrangular pyramid two sections are drawn: one through the diagonals of the bases, the other through the side of the lower base and opposite side of the upper base. The angle between the cutting planes is

equal to a. Find the ratio of the areas of the sections.

62. One cone is inscribed in, and the other is circumscribed about, a regular hexagonal pyramid. Find the difference between the volumes of the circumscribed and inscribed cones if the alti tude of the pyramid is H and the radius of the base of the circumscribed cone is R.

63. Given a ball and a point inside it. Three mutually perpendicular planes intersecting the ball along three circles are passed through this point in an arbitrary way. Prove that the sum of the areas of these three circles is constant, and find this sum if the radius of the ball is R and the distance from the point of intersection of the planes to the centre of the ball is equal to d.

64. In a ball of radius R the diameter AB is drawn. Two straight lines touch the ball at the points A and B and form an angle α ($\alpha < 90°$) between themselves. Taken on these lines are points C and D so that CD touches the ball, and the angle between AB and CD equals φ ($\varphi < 90°$). Find the volume of the tetrahedron $ABCD$.

65. In a tetrahedron two opposite edges are perpendicular, their lengths are a and b, the distance between them is c. Inscribed in the tetrahedron is a cube whose four edges are perpendicular to these two edges of the tetrahedron, exactly two vertices of the cube lying on each face of the tetrahedron. Find the edge of the cube.

66. Two congruent triangles KLM and KLN have a common side KL, $\widehat{KLM} = \widehat{LKN} = \pi/3$, $|KL| = a$, $|LM| = |KN| = 6a$. The planes KLM and KLN are mutually perpendicular. A

ball touches the line segments LM and KN at their midpoints. Find the radius of the ball.

67. A ball of radius R touches all the lateral faces of a triangular pyramid at the midpoints of the sides of its base. The line segment joining the vertex of the pyramid to the centre of the ball is bisected by the point of intersection with the base of the pyramid. Find the volume of the pyramid.

68. A tetrahedron has three right dihedral angles. One of the line segments connecting the midpoints of opposite edges of the tetrahedron is equal to a, and the other to b ($b > a$). Find the length of the greatest edge of the tetrahedron.

69. A right circular cone with vertex S is inscribed in a triangular pyramid $SPQR$ so that the circle of the base of the cone is inscribed in the base PQR of the pyramid. It is known that $\widehat{PSR} = \pi/2$, $\widehat{SQR} = \pi/4$, $\widehat{PSQ} = 7\pi/12$. Find the ratio of the lateral surface area of the cone to the area of the base PQR of the pyramid.

70. The base of the pyramid $ABCDE$ is a parallelogram $ABCD$. None of the lateral faces is an obtuse triangle. On the edge DC there is a point M such that the straight line EM is perpendicular to BC. In addition, the diagonal of the base AC and the lateral edges ED and EB are related as follows: $|AC| \geqslant \frac{5}{4}|EB| \geqslant \frac{5}{3}|ED|$. A section representing an isosceles trapezoid is passed through the vertex B and the midpoint of one of the lateral edges. Find the ratio of the area of the section to the area of the base of the pyramid.

71. A line segment AB of unit length which is a chord of a sphere of radius 1 is at an angle $\pi/3$ to the diameter CD of this sphere. The distance from the end point C of the diameter to the nearer end point A of the chord AB is equal to $\sqrt{2}$. Determine the length of the line segment BD.

72. In a triangular pyramid $ABCD$ the faces ABC and ABD have areas p and q, respectively, and form an angle α between themselves. Find the area of the section of the pyramid passing through the edge AB and the centre of the ball inscribed in the pyramid

73. In a triangular pyramid $ABCD$ a section is passed through the edge AD ($|AD| = a$) and point E (the midpoint of the edge BC). The section makes with the faces ACD and ADB angles respectively equal to α and β. Find the volume of the pyramid if the area of the section ADE is equal to S.

74. $ABCD$ is a regular tetrahedron with edge a. Let M be the centre of the face ADC, and let N be the midpoint of the edge BC. Find the radius of the ball inscribed in the trihedral angle A and touching the straight line MN.

75. The base of a triangular pyramid $ABCD$ is a regular triangle ABC. The face BCD makes an angle of 60° with the plane of the base. The centre of a circle of unit radius which touches the edges AB, AC, and the face BCD lies on the straight line passing through the point D perpendicular to the base. The altitude of the pyramid DH is one-half the side of the base. Find the volume of the pyramid.

76. In a triangular pyramid $SABC$ $|AC| = |AB|$ and the edge SA is inclined to the planes

of the faces ABC and SBC at angles of 45°. It is known that the vertex A and the midpoints of all the edges of the pyramid, except SA, lie on the sphere of radius 1. Prove that the centre of the sphere is located on the edge SA, and find the area of the face ASC.

77. Given a cube $ABCDA_1B_1C_1D_1$ with edge a. Find the radius of the sphere touching the line segments AC_1 and CC_1, the straight lines AB and BC and intersecting the straight lines AC and A_1C_1.

78. A ball touches the plane of the base $ABCD$ of a regular quadrangular pyramid $SABCD$ at the point A, and, besides, it touches the ball inscribed in the pyramid. A cutting plane is passed through the centre of the first ball and the side BC of the base. Find the angle of inclination of this plane to the plane of the base if it is known that the diagonals of the section are perpendicular to the edges SA and SD.

79. Situated on a sphere of radius 2 are three circles of radius 1 each of which touches the other two. Find the radius of the circle which is smaller than the given circles, lies on the given sphere, and touches each of the given circles.

80. In a given rectangular parallelepiped $ABCDA_1B_1C_1D_1$ the lengths of the edges AB, BC, and BB_1 are respectively equal to $2a$, a, and a; E is the midpoint of the edge BC. The vertices M and N of a regular tetrahedron $MNPQ$ lie on the straight line C_1E, the vertices P and Q on the straight line passing through the point B_1 and intersecting the straight line AD at the point F. Find: (a) the length of the line segment

DF; (b) the distance between the midpoints of the line segments MN and PQ.

81. The length of the edge of a cube $ABCDA_1B_1C_1D_1$ is a. The points M and N lie on the line segments BD and CC_1, respectively. The straight line MN makes an angle $\pi/4$ with the plane $ABCD$ and an angle $\pi/6$ with the plane BB_1C_1C. Find: (a) the length of the line segment MN; (b) the radius of the sphere with centre on the line segment MN which touches the planes $ABCD$ and BB_1C_1C.

82. The vertex A of a regular prism $ABCA_1B_1C_1$ coincides with the vertex of a cone; the vertices B and C lie on the lateral surface of this cone, and the vertices B_1 and C_1 on the circle of its base. Find the ratio of the volume of the cone and the prism if $|AA_1| = 2.4\,|AB|$.

83. The length of the edge of a cube $ABCDA_1B_1C_1D_1$ is equal to a. The points P, K, L are midpoints of the edges AA_1, A_1D_1, B_1C_1, respectively; the point Q is the centre of the face CC_1D_1D. The line segment MN with end points on the straight lines AD and KL intersects the line PQ and is perpendicular to it. Find the length of this line segment.

84. In a regular prism $ABCA_1B_1C_1$ the length of a lateral edge and the altitude of the base is equal to a. Two planes are passed through the vertex A: one perpendicular to the straight line AB_1, the other perpendicular to the line AC_1. Passed through the vertex A_1 are also two planes: one perpendicular to the line A_1B, the other perpendicular to the line A_1C. Find the volume of the polyhedron bounded by these four planes and the plane BB_1C_1C.

85. The point O is a common vertex of two congruent cones situated on one side of the plane α so that only one element of each cone (OA for one cone and OB for the other) belongs to the plane α. It is known that the size of the angle between the altitudes of the cones is equal to β, and the size of the angle between the altitude and generatrix of the cone is equal to φ, and $2\varphi < \beta$. Find the size of the angle between the element OA and the plane of the base of the other cone to which the point B belongs.

86. Arranged inside a regular tetrahedron $ABCD$ are two balls of radii $2R$ and $3R$ externally tangent to each other, one ball being inscribed in the trihedral angle of the tetrahedron with vertex at the point A, and the other in the trihedral angle with vertex at the point B. Find the length of the edge of this tetrahedron.

87. In a regular quadrangular pyramid $SABCD$ with base $ABCD$, the side of the base is equal to a, and the angle between the lateral edges and the plane of the base is equal to α. The plane parallel to the diagonal of the base AC and the lateral edge BS cuts the pyramid so that a circle can be inscribed in the section obtained. Determine the radius of this circle.

88. Each edge of a regular tetrahedron is equal to a. A plane P passes through the vertex B and midpoints of the edges AC and AD. A ball touches the straight lines AB, AC, AD and the portion of the plane P enclosed inside the tetrahedron. Find the radius of the ball.

89. In a regular tetrahedron, M and N are midpoints of two opposite edges. The projection of the tetrahedron on a plane parallel to MN

is a quadrilateral having area S one of the angles of which is equal to 60°. Find the surface area of the tetrahedron.

90. In a cube $ABCDA_1B_1C_1D_1$ a point M is taken on AC, and on the diagonal BD_1 of the cube a point N is taken so that $\widehat{NMC} = 60°$, $\widehat{MNB} = 45°$. In what ratios are the line segments AC and BD_1 divided by the points M and N?

91. The base of a right prism $ABCDA_1B_1C_1D_1$ is an isosceles trapezoid $ABCD$ in which AD is parallel to BC, $|AD|/|BC| = n$, $n > 1$. Passed through the edges AA_1 and BC are planes parallel to the diagonal B_1D; and through the edges DD_1 and B_1C_1 planes parallel to the diagonal A_1C. Determine the ratio of the volume of the triangular pyramid bounded by these four planes to the volume of the prism.

92. The side of the base of a regular triangular prism $ABCA_1B_1C_1$ is equal to a. The points M and N are the respective midpoints of the edges A_1B_1 and AA_1. The projection of the line segment BM on the line C_1N is equal to $a/2\sqrt{5}$. Determine the altitude of the prism.

93. Two balls touch each other and the faces of a dihedral angle whose size is α. Let A and B be points at which the balls touch the faces (A and B belong to different balls and different faces). In what ratio is the line segment AB divided by the points of intersection with the surfaces of the balls?

94. The base of a pyramid $ABCD$ is a regular triangle ABC with side of length 12. The edge BD

is perpendicular to the plane of the base and is equal to $10\sqrt{3}$. All the vertices of this pyramid lie on the lateral surface of a right circular cylinder whose axis intersects the edge BD and the plane ABC. Determine the radius of the cylinder.

95. The base of a pyramid is a square $ABCD$ with side a; the lateral edge SC is perpendicular to the plane of the base and is equal to b. M is a point on the edge AS. The points M, B, and D lie on the lateral surface of a right circular cone with vertex at the point A, and the point C in the plane of the base of this cone. Determine the area of the lateral surface of the cone.

96. Inside a right circular cone a cube is arranged so that one of its edges lies on the diameter of the base of the cone; the vertices of the cube not belonging to this edge lie on the lateral surface of the cone; the centre of the cube lies on the altitude of the cone. Find the ratio of the volume of the cone to the volume of the cube.

97. In a triangular prism $ABCA_1B_1C_1$, two sections are passed. One section passes through the edge AB and midpoint of the edge CC_1, the other passing through the edge A_1B_1 and the midpoint of the edge CB. Find the ratio of the length of the line segment of the intersection line of these sections enclosed inside the prism to the length of the edge AB.

98. In the tetrahedron $ABCD$ the edge AB is perpendicular to the edge CD, $\widehat{ACB} = \widehat{ADB}$, the area of the section passing through the edge AB and the midpoint of the edge DC is equal to S, $|DC| = a$. Find the volume of the tetrahedron $ABCD$.

99. Given a regular triangular pyramid $SABC$ (S its vertex). The edge SC of this pyramid coincides with a lateral edge of a regular triangular prism $A_1B_1CA_2B_2S$ (A_1A_2, B_1B_2 and CS are lateral edges, and A_1B_1C is one of the bases). The vertices A_1 and B_1 lie in the plane of the face SAB of the pyramid. What part of the volume of the entire pyramid is the volume of the portion of the pyramid lying inside the prism if the ratio of the length of the lateral edge of the pyramid to the side of its base is equal to $2/\sqrt{3}$?

100. In a frustum of a regular quadrangular pyramid with the lateral edges AA_1, BB_1, CC_1, DD_1, the side of the upper base $A_1B_1C_1D_1$ is equal to 1, and the side of the lower base is equal to 7. The plane passing through the edge B_1C_1 perpendicular to the plane AD_1C separates the pyramid into two parts of equal volume. Find the volume of the pyramid.

101. The base of the prism $ABCA_1B_1C_1$ is a regular triangle ABC with side a. The projection of the prism on the plane of the base is a trapezoid with lateral side AB and area which is twice the area of the base. The radius of the sphere passing through the vertices A, B, A_1, C_1 is equal to a. Find the volume of the prism.

102. Given in a plane is a square $ABCD$ with side a and a point M lying at a distance b from its centre. Find the sum of the volumes of the solids generated by revolving the triangles ABM, BCM, CDM, and DAM about the straight lines AB, BC, CD and DA, respectively.

103. D is the midpoint of the edge A_1C_1 of a regular triangular prism $ABCA_1B_1C_1$. A regular triangular pyramid $SMNP$ is situated so that the plane of its base MNP coincides with the plane ABC, the vertex M lies on the extension of AC and $|CM| = \frac{1}{2}|AC|$, the edge SN passes through the point D, and the edge SP intersects the line segment BB_1. In what ratio is the line segment BB_1 divided by the point of intersection?

104. The centres of three spheres of radii 3, 4, and 6 are situated at the vertices of a regular triangle with side 11. How many planes are there which simultaneously touch all the three spheres?

105. All the plane angles of a trihedral angle $NKLM$ (N the vertex) are right ones. On the face LNM a point P is taken at a distance 2 from the vertex N and at a distance 1 from the edge MN. From some point S situated inside the trihedral angle $NKLM$ a beam of light is directed towards the point P. The beam makes an angle $\pi/4$ with the plane MNK and equal angles with the edges KN and MN. The beam is mirror-reflected from the faces of the angle $NKLM$ first at the point P, then at the point Q, and then at the point R. Find the sum of the lengths of the line segments PQ and QR.

106. The base of a triangular pyramid $ABCD$ is a triangle ABC in which $\hat{A} = \pi/2$, $\hat{C} = \pi/6$, $|BC| = 2\sqrt{2}$. The edges AD, BD, and CD are of the same length. A sphere of radius 1 touches the edges AD, BD, the extension of the edge CD beyond the point D, and the plane ABC. Find the length of the line segment of the tangent drawn from the point A to the sphere.

107. Three balls, among which there are two equal balls, touch a plane P and, besides, pairwise touch one another. The vertex of a right circular cone belongs to the plane P, and its axis is perpendicular to this plane. All the three balls are arranged outside of the cone and each of them touches its lateral surface. Find the cosine of the angle between the generatrix of the cone and the plane P if it is known that in the triangle with vertices at the points of tangency of the balls with the plane one of the angles is equal to $150°$.

108. The volume of the tetrahedron $ABCD$ is equal to 5. Through the midpoints of the edges AD and BC a plane is passed cutting the edge CD at the point M. And the ratio of the lengths of the line segments DM and CM is equal to 2/3. Compute the area of the section of the tetrahedron by the plane if the distance from it to the vertex A is equal to 1.

109. A ball of radius 2 is inscribed in a regular triangular pyramid $SABC$ with vertex S and base ABC; the altitude of the pyramid SK is equal to 6. Prove that there is a unique plane cutting the edges of the base AB and BC at some points M and N, such that $|MN| = 7$, which touches the ball at the point equidistant from the points M and N and intersects the extension of the altitude of the pyramid SK beyond the point K at some point D. Find the length of the line segment SD.

110. All the edges of a triangular pyramid $ABCD$ are tangent to a sphere. Three line segments joining the midpoints of skew edges have the same length. The angle ABC is equal to $100°$. Find the ratio of the altitudes of the pyramid drawn from the vertices A and B.

111. In a pyramid $SABC$ the products of the lengths of the edges of each of the four faces are equal to one and the same number. The length of the altitude of the pyramid dropped from S onto the face ABC is equal to $2\sqrt{\frac{102}{55}}$, and the size of the angle CAB is equal to $\arccos\left(\frac{1}{6}\sqrt{\frac{17}{2}}\right)$. Find the volume of the pyramid $SABC$ if

$|SA|^2 + |SB|^2 - 5|SC|^2 = 60.$

112. Given in a plane P is an isosceles triangle ABC ($|AB| = |BC| = l$, $|AC| = 2a$). A sphere of radius r touches the plane P at point B. Two skew lines pass through the points A and C and are tangent to the ball. The angle between either of these lines and the plane P is equal to α. Find the distance between these lines.

113. The base of a pyramid $ABCEH$ is a convex quadrilateral $ABCE$ which is separated by the diagonal BE into two equivalent triangles. The length of the edge AB is equal to 1, the lengths of the edges BC and CE are equal to each other. The sum of the lengths of the edges AH and EH is equal to $\sqrt{2}$. The volume of the pyramid is 1/6. Find the radius of the sphere having the greatest volume among all the balls housed in the pyramid.

114. In a pyramid $SABC$ a straight line intersecting the edges AC and BS and perpendicular to them passes through the midpoint of the edge BS. The face ASB is equivalent to the face BSC, and the area of the face ASC is twice the area of the face BSC. Inside the pyramid there is a point M, and the sum of the distances from this point to the vertices B and S is equal to the sum of the distances to all the faces of the pyramid. Find the distance from the point M to the vertex B if $|AC| = \sqrt{6}$, $|BS| = 1$.

115. The base of a pyramid is a rectangle with acute angle between the diagonals α ($\alpha < 60°$), its lateral edges are of the same length, and the altitude is h. Situated inside the pyramid is a triangular pyramid whose vertex coincides with the vertex of the first pyramid, and the vertices of the base lie on three sides of the rectangle. Find the volume of the quadrangular pyramid if all the edges of the triangular pyramid are equal to one another, and the lateral faces are equivalent.

116. In a triangular pyramid $SABC$ with base ABC and equal lateral edges, the sum of the dihedral angles with edges SA and SC is equal to 180°. It is known that $|AB| = a$, $|BC| = b$. Find the length of the lateral edge.

117. Given a regular tetrahedron with edge a. A sphere touches three edges of the tetrahedron, emanating from one vertex, at their end points. Find the area of the portion of the spherical surface enclosed inside the tetrahedron.

118. Three circles of radius $\sqrt{2}$ pairwise touching one another are situated on the surface of a sphere of radius 2. The portion of the sphere's surface situated outside of the circles presents two curvilinear triangles. Find the areas of these triangles.

119. Three dihedral angles of a tetrahedron, not belonging to one vertex, are equal to $\pi/2$.

The remaining three dihedral angles are equal to one another. Find these angles.

120. Two balls are inscribed in the lateral surface of a cone and touch each other. A third sphere passes through two circles along which the first two spheres touch the surface of the cone. Prove that the volume of the portion of the third ball situated outside of the cone is equal to the volume of the portion of the cone enclosed between the first two balls inside the cone.

121. A sphere of radius R touches one base of a frustum of a cone and its lateral surface along the circle coinciding with the circle of the other base of the cone. Find the volume of the solid representing a combination of a cone and a ball if the total surface area of this solid is equal to S.

122. Two triangles, a regular one with side a and a right isosceles triangle with legs equal to b, are arranged in space so that their centroids coincide. Find the sum of the squared distances from all the vertices of one of them to all the vertices of the other.

123. In a regular triangular pyramid $SABC$ (S the vertex), E is the midpoint of the slant height of the face SBC, and the points F, L, and M lie on the edges AB, AC, and SC, respectively, and $|AL| = \frac{1}{10}|AC|$. It is known that $EFLM$ is an isosceles trapezoid and the length of its base EF is equal to $\sqrt{7}$. Find the volume of the pyramid.

124. Given a cube $ABCDA_1B_1C_1D_1$ with edge a. The bases of a cylinder are inscribed in the faces $ABCD$ and $A_1B_1C_1D_1$. Let M be a point on the edge AB such that $|AM| = a/3$, N a point on the edge B_1C_1 such that $|NC_1| = a/4$. Through the points C_1 and M there passes a plane touching the bases of the cylinder inscribed in $ABCD$, and through A and N a plane touching the base inscribed in $A_1B_1C_1D_1$. Find the volume of the portion of the cylinder enclosed between the planes.

125. Determine the total surface area of the prism circumscribed about a sphere if the area of its base is equal to S.

126. The centre of sphere α lies on the surface of sphere β. The ratio of the surface area of sphere β lying inside sphere α to the total surface area of sphere α is equal to 1/5. Find the ratio of the radii of spheres α and β.

127. Circumscribed about a ball is a frustum of a cone. The total surface area of this cone is S. Another sphere touches the lateral surface of the cone along the circle of the base of the cone. Find the volume of the frustum of a cone if it is known that the portion of the surface of the second ball contained inside the first ball has an area Q.

128. Circumscribed about a ball is a frustum of a cone whose bases are the great circles of two other balls. Determine the total surface area of the frustum of a cone if the sum of the surface areas of the three balls is equal to S.

129. A section of maximal area is passed through the vertex of a right circular cone. It is known that the area of this section is twice the area of an axial section. Find the vertex angle of the axial section of the cone.

130. Inscribed in a cone is a triangular pyramid $SABC$ (S coincides with the vertex of the cone, A, B, and C lie on the circle of the base of the cone), the dihedral angles at the edges SA, SB, and SC are respectively equal to α, β, and γ. Find the angle between the plane SBC and the plane touching the surface of the cone along the element SC.

131. Three points A, B, and C lying on the surface of a sphere of radius R are pairwise connected by arcs of great circles; the arcs are less than a semicircle. Through the midpoints of the arcs $\smile{AB}$ and $\smile{AC}$ one more great circle is drawn which intersects the continuation of $\smile{BC}$ at the point K. Find the length of the arc $\smile{CK}$ if $|\smile{BC}| = l$ $(l < \pi R)$.

132. Find the volume of the solid generated by revolving a regular triangle with side a about a straight line parallel to its plane and such that the projection of this line on the plane of the triangle contains one of the altitudes of the triangle.

133. Consider the solid consisting of points situated at a distance not exceeding d from an arbitrary point inside a plane figure having a perimeter $2p$ and area S or on its boundary. Find the volume of this solid.

134. Given a triangular pyramid $SABC$. A ball of radius R touches the plane ABC at the point C and the edge SA at the point S. The straight line BS intersects the ball for the second time at the point opposite to the point C. Find the volume of the pyramid $SABC$ if $|BC| = a$, $|SA| = b$.

135. Inside a regular triangular pyramid there is a vertex of a trihedral angle all of whose plane angles are right ones, and the bisectors of the plane angles pass through the vertices of the base. In what ratio is the volume of the pyramid divided by the surface of this angle if each face of the pyramid is separated by it into two equivalent portions?

136. Given a parallelepiped $ABCDA_1B_1C_1D_1$ whose volume is V. Find the volume of the common portion of two tetrahedrons AB_1CD_1 and A_1BC_1D.

137. Two equal triangular pyramids each having volume V are arranged in space symmetrically with respect to the point O. Find the volume of their common portion if the point O lies on the line segment joining the vertex of the pyramid to the centroid of the base and divides this line segment in the ratio: (1) 1 : 1; (2) 3 : 1; (3) 2 : 1; (4) 4 : 1, reckoning from the vertex.

138. A regular tetrahedron of volume V is rotated about the straight line joining the midpoints of its skew edges at an angle α. Find the volume of the common portion of the given and turned tetrahedrons $(0 < \alpha < \pi)$.

139. The edge of a cube is a. The cube is rotated about the diagonal through an angle α. Find the volume of the common portion of the original cube and the cube being rotated.

140. A ray of light falls on a plane mirror at an angle α. The mirror is rotated about the projection of the beam on the mirror through an angle β. By what angle will the reflected ray deflect?

141. Given in space are four points: A, B, C, and D, where $|AB| = |BC| = |CD|$, $\widehat{ABC} =$

$\widehat{BCD} = \widehat{CDA} = a$. Find the angle between the straight lines AC and BD.

142. Given a regular n-gonal prism. The area of its base is equal to S. Two planes cut all the lateral edges of the prism so that the volume of the portion of the prism enclosed between the planes is equal to V. Find the sum of the lengths of the segments of the lateral edges of the prism enclosed between the cutting planes if it is known that the planes have no common points inside the prism.

143. Three successive sides of a plane convex pentagon are equal to 1, 2, and a. Find the two remaining sides of this pentagon if it is known that the pentagon is an orthogonal projection on the plane of regular pentagon. For what values of a does the problem have a solution?

144. Given a cube $ABCDA_1B_1C_1D_1$ in which M is the centre of the face ABB_1A_1, N a point on the edge B_1C_1, L the midpoint of A_1B_1, K the foot of the perpendicular dropped from N on BC_1. In what ratio is the edge B_1C_1 divided by the point N if $\widehat{LMK} = \widehat{MKN}$?

145. In a regular hexagonal pyramid the centre of the circumscribed sphere lies on the surface of the inscribed sphere. Find the ratio of the radii of the circumscribed and inscribed spheres.

146. In a regular quadrangular pyramid, the centre of the circumscribed ball lies on the surface of the inscribed ball. Find the size of the plane angle at the vertex of the pyramid.

147. The base of a quadrangular pyramid $SABCD$ is a square $ABCD$ with side a. Both angles beween opposite lateral faces are equal to α. Find he volume of the pyramid

148. A plane cutting the surface of a triangular yramid divides the medians of faces emanating rom one vertex in the following ratios: $2:1$, $:2$, $4:1$ (as measured from the vertex). In hat ratio does this plane divide the volume of his pyramid?

149. n congruent cones have a common vertex. ach one touches its two neighbouring cones along n element, and all the cones touch the same plane. ind the angle at the vertex of the axial sections f the cones.

150. Given a cube $ABCDA_1B_1C_1D_1$. The plane assing through the point A and touching the all inscribed in the cube cuts the edges A_1B_1 nd A_1D_1 at points K and N. Determine the size f the dihedral angle between the planes AC_1K nd AC_1N

151. Given a tetrahedron $ABCD$. Another etrahedron $A_1B_1C_1D_1$ is arranged so that its ertices A_1, B_1, C_1, D_1 lie respectively in the lanes BCD, CDA, DAB, ABC, and the planes f its faces $A_1B_1C_1$, $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$ ontain the respective vertices D, A, B, and C f the tetrahedron $ABCD$. It is also known that e point A_1 coincides with the centre of gravity f the triangle BCD, and the straight lines BD_1, B_1, and DC_1 bisect the line segments AC, AD, nd AB, respectively. Find the volume of the ommon part of these tetrahedrons if the volume f the tetrahedron $ABCD$ is equal to V.

152. In the tetrahedron $ABCD$: $|BC| = CD| = |DA|$, $|BD| = |AC|$, $|BD| > BC|$, the dihedral angle at the edge AB is

equal to $\pi/3$. Find the sum of the remaining dihedral angles.

153. Given a triangular prism $ABCA_1B_1C_1$. It is known that the pyramids $ABCC_1$, ABB_1C_1, and $AA_1B_1C_1$ are congruent. Find the dihedral angles between the plane of the base and the lateral faces of the prism if its base is a nonisosceles right triangle.

154. In a regular tetrahedron $ABCD$ with edge a, taken in the planes BCD, CDA, DAB, and ABC are the respective points A_1, B_1, C_1, and D_1 so that the line A_1B_1 is perpendicular to the plane BCD, B_1C_1 is perpendicular to the plane CDA, C_1D_1 is perpendicular to the plane DAB, and finally, D_1A_1 is perpendicular to the plane ABC. Find the volume of the tetrahedron $A_1B_1C_1D_1$.

155. n congruent balls of radius R touch internally the lateral surface and the plane of the base of a cone, each ball touching two neighbouring balls; n balls of radius $2R$ are arranged in a similar way touching externally the lateral surface of the cone. Find the volume of the cone.

156. Given a cube $ABCDA_1B_1C_1D_1$. The points M and N are taken on the line segments AA_1 and BC_1 so that the line MN intersects the line B_1D. Find

$$\frac{|BC_1|}{|BN|}-\frac{|AM|}{|AA_1|}.$$

157. It is known that all the faces of a tetrahedron are similar triangles, but not all of them are congruent. Besides, any two faces have at least one pair of congruent edges not counting a common edge. Find the volume of this tetrahedron if the lengths of two edges lying in one face are equal to 3 and 5.

158. Given three mutually perpendicular lines, the distance between any two of them being equal to a. Find the volume of the parallelepiped whose diagonal lies on one line, and the diagonals of two adjacent faces on two other lines.

159. The section of a regular quadrangular pyramid by some cutting plane presents a regular pentagon with side a. Find the volume of the pyramid.

160. Given a triangle ABC whose area is S, and the radius of the circumscribed circle is R. Erected to the plane of the triangle at the vertices A, B, and C are three perpendiculars, and points A_1, B_1, and C_1 are taken on them so that the line segments AA_1, BB_1, CC_1 are equal in length to the respective altitudes of the triangle dropped from the vertices A, B, and C. Find the volume of the pyramid bounded by the planes A_1B_1C. A_1BC_1, AB_1C_1, and ABC.

Section 2

Problems on Proof

161. Do the altitudes intersect at one point in any tetrahedron?

162. Is there a triangular pyramid such that the feet of all the altitudes lie outside the corresponding faces?

163. Prove that a straight line making equal angles with three intersecting lines in a plane is perpendicular to this plane.

164. What regular polygons can be obtained when a cube is cut by a plane?

165. Prove that the sum of plane angles of a trihedral angle is less than 2π, and the sum of dihedral angles is greater than π.

166. Let the plane angles of a trihedral angle be equal to α, β, and γ, and the opposite dihedral angles to A, B, and C, respectively. Prove that the following equalities hold true:

$$(1)\quad \frac{\sin\alpha}{\sin A}=\frac{\sin\beta}{\sin B}=\frac{\sin\gamma}{\sin C}$$

(theorem of sines for a trihedral angle),

$$(2)\quad \cos\alpha=\cos\beta\cos\gamma+\sin\beta\sin\gamma\cos A$$

(first theorem of cosines for a trihedral angle),

$$(3)\quad \cos A=-\cos B\cos C+\sin B\sin C\cos\alpha$$

(second theorem of cosines for a trihedral angle).

167. Prove that if all the plane angles of a trihedral angle are obtuse, then all the dihedral angles are also obtuse.

168. Prove that if in a trihedral angle all the dihedral angles are acute, then all the plane angles are also acute.

169. Prove that in an arbitrary tetrahedron there is a trihedral angle all plane angles of which are acute.

170. Prove that in an arbitrary polygon all faces of which are triangles there is an edge such that all the plane angles adjacent to it are acute.

171. Prove that a trihedral prismatic surface can be cut by a plane in a regular triangle.

172. In a triangular pyramid all the plane angles at the vertex A are right angles, the edge AB is equal to the sum of two other edges emanating from A. Prove that the sum of the plane angles at the vertex B is equal to $\pi/2$.

173. Can any trihedral angle be cut by a plane in a regular triangle?

174. Find the plane angles at the vertex of a trihedral angle if it is known that any of its sections by a plane is an acute triangle.

175. Prove that in any tetrahedron there is a vertex such that from the line segments equal to the lengths of the edges emanating from this vertex a triangle can be constructed.

176. Prove that any tetrahedron can be cut by a plane into two parts so that the obtained pieces can be brought together in a different way to form the same tetrahedron.

177. Find the plane angles at the vertex of a trihedral angle if it is known that there exists another trihedral angle with the same vertex whose edges lie in the planes forming the faces of the given angle and are perpendicular to the opposite edges of the given angle.

178. A straight line l makes acute angles α, β, and γ with three mutually perpendicular lines. Prove that $\alpha + \beta + \gamma < \pi$.

179. Prove that the sum of the angles made by the edges of a trihedral angle with opposite faces is less than the sum of its plane angles.

Prove also that if the plane angles of a trihedral angle are acute, then the sum of the angles made by its edges with opposite faces is greater than one half the sum of the plane angles. Does the last statement hold for an arbitrary trihedral angle?

180. Prove that the sum of four dihedral angles of a tetrahedron (excluding any two opposite angles) is less than 2π, and the sum of all dihedral angles of a tetrahedron lies between 2π and 3π.

181. From an arbitrary point of the base of a regular pyramid a perpendicular is erected. Prove that the sum of the line segments from the foot of the perpendicular to the intersection with the lateral faces or their extensions is constant.

182. Prove that if x_1, x_2, x_3, x_4 are distances from an arbitrary point inside a tetrahedron to its faces, and h_1, h_2, h_3, h_4 are the corresponding altitudes of the tetrahedron, then

$$\frac{x_1}{h_1}+\frac{x_2}{h_2}+\frac{x_3}{h_3}+\frac{x_4}{h_4}=1.$$

183. Prove that the plane passing through the midpoints of two skew edges of a tetrahedron cuts it into two parts of equal volumes.

184. Prove that if the base of a pyramid $ABCD$ is a regular triangle ABC, and $\widehat{DAB} = \widehat{DBC} = \widehat{DCA}$, then $ABCD$ is a regular pyramid.

185. Let a and a_1, b and b_1, c and c_1 be pairs of opposite edges of a tetrahedron, and let α, β, and γ be the respective angles between them (α, β, and γ do not exceed 90°). Prove that one of the three numbers $aa_1 \cos \alpha$, $bb_1 \cos \beta$, and $cc_1 \cos \gamma$ is the sum of the other two.

186. In a tetrahedron $ABCD$ the edges DA, DB, and DC are equal to the corresponding altitudes of the triangle ABC (DA is equal to the altitude drawn from the vertex A, and so forth). Prove that a sphere passing through three vertices of the tetrahedron intersects the edges emanating from the fourth vertex at three points which are the vertices of a regular triangle.

187. Given a quadrangular pyramid $MABCD$ whose base is a convex quadrilateral $ABCD$. A plane cuts the edges MA, MB, MC, and MD at points K, L, P, and N, respectively. Prove that the following relationship is fulfilled:

$$S_{BCD}\frac{|MA|}{|MK|}+S_{ADB}\frac{|MC|}{|MP|}$$
$$=S_{ABC}\frac{|MD|}{|MN|}+S_{ACD}\frac{|MB|}{|ML|}.$$

188. From an arbitrary point in space perpendiculars are dropped on the faces of a given cube. The six line segments thus obtained are diagonals of six cubes. Prove that six spheres each of which touches all the edges of the respective cube have a common tangent line.

189. Given three parallel lines; A, B, and C are fixed points on these lines. Let M, N, and L be the respective points on the same lines situated on one side of the plane ABC. Prove that if: (a) the sum of the lengths of the line segments AM, BN, and CL is constant, or (b) the sum of the areas of the trapezoids $AMNB$, $BNLC$, and $CLMA$ is constant, then the plane MNL passes through a fixed point.

190. The sum of the lengths of two skew edges of a tetrahedron is equal to the sum of the lengths of two other skew edges. Prove that the sum of the dihedral angles whose edges are the first pair of edges is equal to the sum of the dihedral angles whose edges are represented by the second pair of the edges of the tetrahedron.

191. Let O be the centre of a regular tetrahedron. From an arbitrary point M taken on one of the faces of the tetrahedron perpendiculars are dropped on its three remaining faces, K, L, and N being the feet of these perpendiculars. Prove that the line OM passes through the centre of gravity of the triangle KLN.

192. In a tetrahedron $ABCD$, the edge CD is perpendicular to the plane ABC, M is the midpoint of DB, and N is the midpoint of AB; K is a point on CD such that $|CK| = \frac{1}{3}|CD|$. Prove that the distance between the lines BK and CN is equal to that between the lines AM and CN.

193. Taken in the plane of one of the lateral faces of a regular quadrangular pyramid is an arbitrary triangle. This triangle is projected on the base of the pyramid, and the obtained triangle is again projected on a lateral face adjacent to the given one. Prove that the last projecting yields a triangle which is similar to the originally taken.

194. In a tetrahedron $ABCD$, an arbitrary point A_1 is taken in the face BCD. An arbitrary plane is passed through the vertex A. The straight lines passing through the vertices B, C. and D parallel to the line AA_1 pierce this plane at points B_1, C_1, and D_1. Prove that the volume of the tetrahedron $A_1B_1C_1D_1$ is equal to the volume of the tetrahedron $ABCD$.

195. Given a tetrahedron $ABCD$. In the planes determining its faces, points A_1, B_1, C_1, D_1 are taken so that the lines AA_1, BB_1, CC_1, DD_1 are parallel to one another. Find the ratio of the volumes of the tetrahedrons $ABCD$ and $A_1B_1C_1D_1$.

196. Let D be one of the vertices of a tetrahedron, M its centre of gravity, O the centre of the circumscribed ball. It is known that the points D, M and the median points of the faces containing D lie on the surface of the same sphere. Prove that the lines DM and OM are mutually perpendicular.

197. Prove that no solid in space can have even number of symmetry axes.

198. Given a circle and a point A in space. Let B be the projection of A on the plane of the given circle, D an arbitrary point of the circle. Prove that the projections of B on AD lie on the same circle.

199. The base of a pyramid $ABCDE$ is a quadrilateral $ABCD$ whose diagonals AC and BD are mutually perpendicular and intersect at point M. The line segment EM is the altitude of the pyramid. Prove that the projections of the point M on the lateral faces of the pyramid lie in one plane.

200. Prove that if the straight line passing through the centre of gravity of the tetrahedron $ABCD$ and the centre of the sphere circumscribed about it intersects the edges AB and CD, then $|AC| = |BD|$, $|AD| = |BC|$.

201. Prove that if the straight line passing through the centre of gravity of the tetrahedron $ABCD$ and the centre of the sphere inscribed in it intersects the edges AB and CD, then $|AC| = |BD|$, $|AD| = |BC|$.

202. Given a cube $ABCDA_1B_1C_1D_1$. Passed through the vertex A is a plane touching the sphere inscribed in the cube. Let M and N be the points of intersection of this plane and the lines A_1B and A_1D. Prove that the line MN is tangent to the ball inscribed in the cube.

203. Prove that for a tetrahedron in which all the plane angles at one of its vertex are right angles the following statement holds true: the sum of the squared areas of rectangular faces is equal to the squared area of the fourth face (Pythagorean theorem for a rectangular tetrahedron).

204. Prove that the sum of the squared projections of the edges of a cube on an arbitrary plane is constant.

205. Prove that the sum of the squared projections of the edges of a regular tetrahedron on an arbitrary plane is constant.

206. Two bodies in space move in two straight lines with constant and unequal velocities. Prove that there is a fixed circle in space such that the ratio of distances from any point of this circle to the bodies is constant and is equal to the ratio of their velocities.

207. Given a ball and two points A and B outside it. Two intersecting tangents to the ball are drawn from the points A and B. Prove that the point of their intersection lies in one of the two fixed planes.

208. Three balls touch the plane of a given triangle at its vertices and are tangent to one another. Prove that if the triangle is scalene, then there exist two balls touching the three

given balls and the plane of the triangle, and if r and ρ ($\rho > r$) are the radii of these balls and R is the radius of the circle circumscribed about the triangle, then $\frac{1}{r}-\frac{1}{\rho}=\frac{2\sqrt{3}}{R}$.

209. Given a tetrahedron $ABCD$. One ball touches the edges AB and CD at points A and C, the other at points B and D. Prove that the projections of AC and BD on the straight line passing through the centres of these balls are equal.

210. Is there a space pentagon such that a line segment joining any two nonadjacent vertices intersects the plane of the triangle formed by the remaining three vertices at an interior point of this triangle?

211. Prove that a pentagon with equal sides and angles is plane.

212. Given a parallelepiped $ABCDA_1B_1C_1D_1$ whose diagonal AC_1 is equal to d and its volume to V. Prove that from the line segments equal to the distances from the vertices A_1, B, and D to the diagonal AC_1 it is possible to construct a triangle, and that if s is the area of this triangle, then $V = 2ds$.

213. Given a tetrahedron $ABCD$ in which A_1, B_1, C_1, D_1 are the median points of the faces BCD, CDA, DAB, and ABC. Prove that there is a tetrahedron $A_2B_2C_2D_2$ in which the edges A_2B_2, B_2C_2, C_2D_2 and D_2A_2 are equal and parallel to the line segments AA_1, BB_1, CC_1, and DD_1, respectively. Find the volume of the tetrahedron $A_2B_2C_2D_2$ if the volume of the tetrahedron $ABCD$ is equal to V.

214. Given a tetrahedron. Prove that there is another tetrahedron $KLMN$ whose edges KL,

LM, MN, and NK are perpendicular to the corresponding faces of the given tetrahedron, and their lengths are numerically equal to the areas of these faces. Find the volume of the tetrahedron $KLMN$ if the volume of the given tetrahedron is equal to V.

215. Given three intersecting spheres. Three chords belonging to different spheres are drawn through a point, situated on the chord common for all the three spheres. Prove that the end points of the three chords lie on one and the same sphere.

216. A tetrahedron $ABCD$ is cut by a plane perpendicular to the radius of the circumscribed sphere drawn towards the vertex D. Prove that the vertices A, B, C and the points of intersection of the plane with the edges DA, DB, DC lie on one and the same sphere.

217. Given a sphere, a circle on the sphere, and a point P not belonging to the sphere. Prove that the other points of intersection of the lines, connecting the point P and the points on the given circle, form a circle with the surface of the sphere.

218. Prove that the line of intersection of two conical surfaces with parallel axes and equal angles of axial sections is a plane curve.

219. Taken on the edges AB, BC, CD, and DA of the tetrahedron $ABCD$ are points K, L, M, and N situated in one and the same plane. Let P be an arbitrary point in space. The lines PK, PL, PM, and PN intersect once again the circles circumscribed about the triangles PAB, PBC, PCD, and PDA at the points Q, R, S, and T, respectively. Prove that the points P, Q, R, S, and T lie on the surface of a sphere.

220. Prove that the edges of a tetrahedral angle are elements of a cone whose vertex coincides with the vertex of this angle if and only if the sums of the opposite dihedral angles of the tetrahedral angle are equal to each other.

221. Given a hexagon all faces of which are quadrilaterals. It is known that seven of its eight vertices lie on the surface of one sphere. Prove that the eighth vertex also lies on the surface of the same sphere.

222. Taken on each edge of a tetrahedron is an arbitrary point different from the vertex of the tetrahedron. Prove that four spheres each of which passes through one vertex of the tetrahedron and three points taken on the edges emanating from this vertex intersect at one point.

Section 3

Problems on Extrema. Geometric Inequalities

223. Given a dihedral angle. A straight line l lies in the plane of one of its faces. Prove that the angle between the line l and the plane of the other face is maximal when l is perpendicular to the edge of this dihedral angle.

224. In a convex quadrihedral angle, each of the plane angles is equal to 60°. Prove that the angles between opposite edges ncanot be all acute or all obtuse.

225. The altitude of a frustum of a pyramid is equal to h, and the area of the midsection is S.

What is the range of change of the volume of this pyramid?

226. Find the greatest value of the volume of the tetrahedron inscribed in a cylinder the radius of whose base is R and the altitude is h.

227. The base of a rectangular parallelepiped $ABCDA_1B_1C_1D_1$ is a square $ABCD$. Find the greatest possible size of the angle between the line BD_1 and the plane BDC_1.

228. In a regular quadrangular prism $ABCDA_1B_1C_1D_1$ the altitude is one half the side of the base. Find the greatest size of the angle A_1MC_1, where M is a point on the edge AB.

229. The length of the edge of the cube $ABCDA_1B_1C_1D_1$ is equal to 1. On the extension of the edge AD, a point M is chosen for the point D so that $|AM| = 2\sqrt{2/5}$. Point E is the midpoint of the edge A_1B_1, and point F is the midpoint of the edge DD_1. What is the greatest value that can be attained by the ratio $|MP|/|PQ|$, where the point P lies on the line segment AE, and the point Q on the line segment CF?

230. The length of the edge of the cube $ABCDA_1B_1C_1D_1$ is equal to a. Points E and F are the midpoints of the edges BB_1 and CC_1, respectively. The triangles are considered whose vertices are the points of intersection of the plane parallel to the plane $ABCD$ with the lines AC_1, CE, and DF. Find the smallest value of the areas of the triangles under consideration.

231. Inscribed in a regular quadrangular pyramid with side of the base and altitude equal to 1 (each) is a rectangular parallelepiped whose base is in the plane of the base of the pyramid, and

the vertices of the opposite face lie on the lateral surface of the pyramid. The area of the base of the parallelepiped is equal to s. What is the range of variation of the length of the diagonal of the parallelepiped?

232. The bases of a frustum of a pyramid are regular triangles ABC and $A_1B_1C_1$ 3 cm and 2 cm on a side, respectively. The line segment joining the vertex C_1 to the centre O of the base ABC is perpendicular to the bases; $|C_1O| = 3$. A plane is passed through the vertex B and midpoints of the edges A_1B_1 and B_1C_1. Consider the cylinders situated inside the polyhedron $ABCA_1MNC_1$ with bases in the face A_1MNC_1. Find: (a) the greatest value of the volumes of such cylinders with a given altitude h; (b) the maximal value of the volume among all cylinders under consideration.

233. All edges of a regular triangular prism $ABCA_1B_1C_1$ have an equal length a. Consider the line segments with end points on the diagonals BC_1 and CA_1 of the lateral faces parallel to the plane ABB_1A_1. Find the minimal length of such line segments.

234. Given a trihedral angle and a point inside it through which a plane is passed. Prove that the volume of the tetrahedron formed by the given angle and the plane will be minimal if the given point is the centre of gravity of the triangle which is the section of the trihedral angle by the plane.

235. The surface area of a spherical segment is equal to S (the spherical part of the segment is considered here). Find the greatest volume of this segment.

236. A cube with edge a is placed on a plane.

A light source is situated at a distance b ($b > a$) from the plane. Find the smallest area of the shadow thrown by the cube onto the plane.

237. Given a convex central-symmetric polyhedron. Consider the sections of this polyhedron parallel to the given plane. Check whether the following statements are true:

(1) the greatest area is possessed by the section passing through the centre;

(2) for each section consider the circle of smallest radius containing this section. Is it true that to the greatest radius of such a circle there corresponds the section passing through the centre of the polyhedron?

238. What is the smallest value which can be attained by the ratio of the volumes of the cone and cylinder circumscribed about the same ball?

239. Two cones have a common base and are arranged on different sides of it. The radius of the base is r, the altitude of one cone is h, of the other H ($h \leqslant H$). Find the maximal distance between two elements of these cones.

240. Given a cube $ABCDA_1B_1C_1D_1$ with edge a. Find the radius of the smallest ball which touches the straight lines AB_1, B_1C, CD, and DA.

241. The diagonal of a cube whose edge is equal to 1 lies on the edge of a dihedral angle of size α ($\alpha < 180°$). What is the range of variation of the volume of the portion of the cube enclosed inside this angle?

242. The lengths of the edges of a rectangular parallelepiped are equal to a, b, and c. What is the greatest value of the area of an orthogonal projection of this parallelepiped on a plane?

243. The length of each of five edges of a tetrahedron is less than unity. Prove that the volume of the tetrahedron is less than 1/8.

244. The vertex E of the pyramid $ABCE$ is found inside the pyramid $ABCD$. Check whether the following statements are true:

(1) the sum of the lengths of the edges AE, BE, and CE is less than that of the edges AD, BD, and CD;

(2) at least one of the edges AE, BE, CE is shorter than the corresponding edge AD, BD, or CD?

245. Let r and R be the respective radii of the balls inscribed in, and circumscribed about, a regular quadrangular pyramid. Prove that

$$\frac{R}{r} \geqslant \sqrt{2}+1$$

246. Let R and r be the respective radii of the balls inscribed in, and circumscribed about, a tetrahedron. Prove that $R \geqslant 3r$.

247. Two opposite edges of a tetrahedron have lengths b and c, the length of the remaining edges being equal to a. What is the smallest value of the sum of distances from an arbitrary point in space to the vertices of this tetrahedron?

248. Given a frustum of a cone in which the angle between the generatrix and greater base is equal to 60°. Prove that the shortest path over the surface of the cone between a point on the boundary of one base and the diametrically opposite point of the other base has a length of $2R$, where R is the radius of the greater base.

249. Let **a**, **b**, and **c** be three arbitrary vectors. Prove that

$$|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| + |\mathbf{a}+\mathbf{b}+\mathbf{c}| \geqslant |\mathbf{a}+\mathbf{b}| + |\mathbf{b}+\mathbf{c}| + |\mathbf{c}+\mathbf{a}|.$$

250. Given a cube $ABCDA_1B_1C_1D_1$ with edge a. Taken on the line AA_1 is a point M, and on the line BC a point N so that the line MN intersects the edge C_1D_1. Find the smallest value of the quantity $|MN|$.

251. The base of a quadrangular pyramid is a rectangle one side of which is equal to a, the length of each lateral edge of the pyramid is equal to b. Find the greatest value of the volume of such pyramids.

252. Given a cube $ABCDA_1B_1C_1D_1$ with edge a. Find the length of the shortest possible segment whose end points are situated on the lines AB_1 and BC_1 making an angle of 60° with the plane of the face $ABCD$.

253. Three equal cylindrical surfaces of radius R with mutually perpendicular axes touch one another pairwise.

(a) What is the radius of the smallest ball touching these cylindrical surfaces?

(b) What is the radius of the greatest cylinder touching the three given cylindrical surfaces, whose axis passes inside the triangle with vertices at the points of tangency of the three given cylinders?

254. Two vertices of a tetrahedron are situated on the surface of the sphere of radius $\sqrt{10}$, and two other vertices on the surface of the sphere of radius 2 which is concentric with the first one. What is the greatest volume of such tetrahedrons?

255. Two trihedral angles are arranged so that the vertex of one of them is equidistant from the faces of the other and vice versa; the distance between the vertices is equal to a. What is the minimal volume of the hexahedron bounded by the faces of these angles if all the plane angles of one of them are equal to 60° (each), and those of the other to 90° (each)?

256. What is the greatest volume of the tetrahedron $ABCD$ all vertices of which lie on the surface of a sphere of radius 1, and the edges AB, BC, CD, and DA are seen from the centre of the sphere at an angle of 60°?

257. Given a regular tetrahedron with edge a. Find the radius of such a ball with centre at the centre of the tetrahedron for which the sum of the volumes of the part of the tetrahedron found outside of the ball and the part of the sphere outside of the tetrahedron reaches its smallest value.

258. Prove that among triangular pyramids with a given base and equal altitudes the smallest lateral surface is possessed by the one whose vertex is projected into the centre of the circle inscribed in the base.

259. Given a cube with edge a. Let N be a point on the diagonal of a lateral face, M a point on the circle found in the plane of the base having its centre at the centre of the base and radius $(5/12)a$. Find the least value of the quantity $|MN|$.

260. (a) The base of the pyramid $SABC$ is a triangle ABC in which $\widehat{BAC} = A$, $\widehat{CBA} = B$, the radius of the circle circumscribed about it is equal to R. The edge SC is perpendicular to the plane ABC. Find $|SC|$ if it is known that $1/\sin\alpha + 1/\sin\beta - 1/\sin\gamma = 1$, where α, β, and γ are angles made by the edges SA, SB, and SC with the planes of the faces SBC, SAC, and SAB, respectively.

(b) Let α, β, and γ be angles made by the edges of a trihedral angle with the planes of opposite faces. Prove that $1/\sin\alpha + 1/\sin\beta - 1/\sin\gamma \geqslant 1$.

261. Can a regular tetrahedron with edge 1 pass through a circular hole of radius: (a) 0.45; (b) 0.44? The thickness of the hole may be neglected.

Section 4

Loci of Points

262. Prove that in an arbitrary trihedral angle the bisectors of two plane angles and the angle adjacent to the third plane angle lie in one plane.

263. Prove that if the lateral surface of a cylinder is cut by an inclined plane, and then it is cut along an element and developed on a plane, the line of intersection will represent a sinusoid.

264. Given on the surface of a cone is a line different from an element and such that any two points of this line can be connected with an arc belonging to this line and representing a line segment on the development. How many points of self-intersection has this line if the angle of the axial section of the cone is equal to α?

265. Three mutually perpendicular lines pass through the point O. A, B, and C are points on these lines such that

$$|OA| = |OB| = |OC|.$$

Let l be an arbitrary line passing through O; A_1, B_1, and C_1 points symmetric to the points A, B, and C with respect to l. Through A_1, B_1, and C_1 three planes are drawn perpendicular to the lines OA, OB, and OC, respectively. Find the locus of points of intersection of these planes.

266. Find the locus of the midpoints of line segments parallel to a given plane whose end points lie on two skew lines.

267. Given three pairwise skew lines. Find:

(a) the locus of centres of gravity of triangles ABC with vertices on these lines;

(b) the locus of centres of gravity of triangles ABC with vertices on these lines whose planes are parallel to a given plane.

268. Three pairwise skew lines l_1, l_2, l_3 are perpendicular to one and the same straight line and intersect it. Let N and M be two points on the lines l_1 and l_2 such that the line NM intersects the line l_3. Find the locus of midpoints of line segments NM.

269. Given in space are several arbitrary lines and a point A. Through A a straight line is

drawn so that the sum of the cosines of the acute angles made by this line with the given ones is equal to a given number. Find the locus of such lines.

270. Given a triangle ABC and a straight line l. A_1, B_1, and C_1 are three arbitrary points on the line l. Find the locus of centres of gravity of triangles with vertices at the midpoints of the line segments AA_1, BB_1, CC_1.

271. Given a straight line l and a point A. Through A an arbitrary line is drawn which is skew with l. Let MN be a common perpendicular to this line and to l (M lies on the line passing through A). Find the locus of points M.

272. Two spheres α and β touch a third sphere ω at points A and B. A point M is taken on the sphere α, the line MA pierces the sphere ω at point N, and the line NB pierces the sphere β at point K. Find the locus of such points M for which the line MK touches the sphere β.

273. Given a plane and two points on one side of it. Find the locus of centres of spheres passing through these points and touching the plane.

274. Find the locus of midpoints of common tangents to two given spheres.

275. Two lines l_1 and l_2 touch a sphere. Let M and N be points on l_1 and l_2 such that the line MN also touches the same sphere. Find the locus of points of tangency of the line MN with this sphere.

276. Given in space are a point O and two straight lines. Find the locus of points M such that the sum of projections of the line segment

OM on the given lines is a constant quantity.

277. Given in space are two straight lines and a point A on one of them; passed through the given lines are two planes making a right dihedral angle. Find the locus of projections of the point A on the edge of this angle.

278. Given three intersecting planes having no common line. Find the locus of points such that the sum of distances from these points to the given planes is constant.

279. Given a triangle ABC. On the straight line perpendicular to the plane ABC and passing through A an arbitrary point D is taken. Find the locus of points of intersection of the altitudes of triangles DBC.

280. Given three intersecting planes and a straight line l. Drawn through a point M in space is a line parallel to l and piercing the given planes at points A, B, and C. Find the locus of points M such that the sum $|AM| + |BM| + |CM|$ is constant.

281. Given a triangle ABC. Find the locus of points M such that the straight line joining the centre of gravity of the pyramid $ABCM$ to the centre of the sphere circumscribed about it intersects the edges AC and BM.

282. A trihedral angle is cut by two planes parallel to a given plane. Let the first plane cut the edges of the trihedral angle at points A, B, and C, and the second at points A_1, B_1, and C_1 (identical letters denote points belonging to one and the same edge). Find the locus of points of intersection of the planes ABC_1, AB_1C, and A_1BC.

283. Given a plane quadrilateral $ABCD$. Find the locus of points M such that the lateral surface of the pyramid $ABCDM$ can be cut by a plane so that the section thus obtained is: (a) a rectangle, (b) a rhombus, (c) a square; (d) in the preceding case find the locus of centres of squares.

284. Given a plane triangle ABC. Find the locus of points M in space such that the straight line connecting the centre of the sphere circumscribed about $ABCM$ with G as the centre of gravity of the tetrahedron $ABCM$ is perpendicular to the plane AMG.

285. A circle of constant radius displaces touching the faces of a trihedral angle all the plane angles of which are equal to 90° (each). Find the locus of centres of these circles.

286. A spider sits in one of the vertices of a cube whose edge is 1 cm long. It crawls over the surface of the cube with a speed of 1 cm/s. Find the locus of points on the surface of the cube such that can be reached by the spider in two seconds.

287. Given a trihedral angle each of whose plane angles is equal to 90°, O is the vertex of this angle. Consider all possible polygonal lines of length a beginning at the point O and such that any plane parallel to one of the faces of the angle cuts this polygonal line not more than at one point. Find the locus of end points of this polygonal line.

288. Given a ball with centre O. Let $ABCD$ be the pyramid circumscribed about it for which the following inequalities are fulfilled: $|OA| \geqslant |OB| \geqslant |OC| \geqslant |OD|$. Find the locus of points A, B, C, and D.

289. Given a triangle ABC. Find the locus of

points M in space such that from the line segments MA, MB, and MC a right triangle can be formed.

290. On the surface of the Earth there are points the geographical latitude of which is equal to their longitude. Find the locus of the projections of all these points on the plane of the equator.

291. Given a right circular cone and a point A outside it found at a distance numerically equal to the altitude of the cone from the plane of its base. Let M be a point on the cone such that a beam of light emanating from A towards M, being mirror-reflected by the surface of the cone, will be parallel to the plane of the base. Find the locus of projections of points M on the plane of the base of the cone.

292. Drawn arbitrarily through a fixed point P inside a ball are three mutually perpendicular rays piercing the surface of the ball at points A, B, and C. Prove that the median point of the triangle ABC and the projection of the point P on the plane ABC describe one and the same spherical surface.

293. Given a trihedral angle with vertex O and a point N. An arbitrary sphere passes through O and N and intersects the edges of the trihedral angle at points A, B, and C. Find the locus of centres of gravity of triangles ABC.

An Arbitrary Tetrahedron

294. Given an arbitrary tetrahedron and a point N. Prove that six planes each of which passes through one edge of the tetrahedron and is parallel to the straight line joining N to the midpoint of the opposite edge intersect at one point.

295. Prove that six planes each of which passes through the midpoint of one edge of the tetrahedron and is perpendicular to the opposite edge intersect at one point (*Monge's point*).

296. Prove that if Monge's point lies in the plane of some face of a tetrahedron, then the foot of the altitude dropped on this face is found on the circle described about it (see the preceding problem).

297. Prove that the sum of squared distances from an arbitrary point in space to the vertices of a tetrahedron is equal to the sum of squared distances between the midpoints of opposite edges and quadruple square of the distance from the point to the centre of gravity of the tetrahedron.

298. Prove that there are at least five and at most eight spheres in an arbitrary tetrahedron each of which touches the planes of all its faces.

299. $ABCD$ is a three-dimensional quadrilateral (A, B, C, and D do not lie in one plane). Prove that there are at least eight balls touching the lines AB, BC, CD, and DA. Prove also that if the sum of some two sides of the given quadrilateral is equal to the sum of two other sides, then there is an infinitude of such balls.

300. Prove that the product of the lengths of two opposite edges of a tetrahedron divided by the product of the sines of the dihedral angles of the tetrahedron corresponding to these edges is constant for a given tetrahedron (theorem of sines).

301. Let S_i, R_i, l_i $(i = 1, 2, 3, 4)$ denote respectively the areas of faces, the radii of the circles circumscribed about these faces, and the distances from the centres of these circles to the opposite vertices of a tetrahedron. Prove that for the vertices of the tetrahedron the following formula is valid:

$$V = \frac{1}{3}\sqrt{\frac{1}{2}\sum_{i=1}^{4} S_i^2 (l_i^2 - R_i^2)}.$$

302. Given an arbitrary tetrahedron. Prove that there exists a triangle whose sides are numerically equal to the products of the lengths of the opposite sides of the tetrahedron. Let S denote the area of this triangle, V the volume of the tetrahedron, R the radius of the sphere circumscribed about it. Then the following equality takes place: $S = 6VR$ (*Crelle's formula*).

303. Let a and b denote the lengths of two skew edges of a tetrahedron, α and β the sizes of the corresponding dihedral angles. Prove that the expression

$$a^2 + b^2 + 2ab \cot \alpha \cot \beta$$

is independent of the choice of the edges (*Bretschneider's theorem*).

An Equifaced Tetrahedron

304. A tetrahedron is said to be *equifaced* if all of its faces are congruent triangles or, which is the same, if opposite edges of the tetrahedron are pairwise equal. Prove that for a tetrahedron to be equifaced, it is necessary and sufficient that any of the following conditions be fulfilled:

(a) the sums of plane angles at any of the three vertices of a tetrahedron are equal to 180°;

(b) the sums of plane angles at some two vertices of a tetrahedron are equal to 180°, and, besides, some two opposite edges are equal;

(c) the sum of plane angles at some vertex of a tetrahedron is equal to 180°, and, besides, the tetrahedron has two pairs of equal opposite edges;

(d) the following equality is fulfilled $\widehat{ABC} = \widehat{ADC} = \widehat{BAD} = \widehat{BCD}$, where $ABCD$ is a given tetrahedron;

(e) all the faces are equivalent;

(f) the centres of the inscribed and circumscribed spheres coincide;

(g) the line segments joining the midpoints of opposite edges are perpendicular;

(h) the centre of gravity coincides with the centre of the circumscribed sphere;

(i) the centre of gravity coincides with the centre of the inscribed sphere.

305. Prove that the sum of cosines of the dihedral angles of a tetrahedron is positive and does not exceed 2, the equality of this sum to 2 is characteristic only of equifaced tetrahedrons.

306. The sum of the plane angles of a trihedral angle is equal to 180°. Find the sum of the cosines of the dihedral angles of this trihedral angle.

307. Prove that for an equifaced tetrahedron

(a) the radius of the inscribed ball is half the radius of the ball which touches one face of the tetrahedron and the extensions of three other faces (such ball is called *externally inscribed*);

(b) the centres of four externally inscribed balls are the vertices of a tetrahedron congruent to the given one.

308. Let h denote the altitude of an equifaced tetrahedron, h_1 and h_2 the line segments into which one of the altitudes of a face is divided (by the point of intersection of the altitudes of this face). Prove that $h^2 = 4h_1h_2$. Prove also that the foot of the altitude of the tetrahedron and the point of intersection of the altitudes of the face on which this altitude is dropped are symmetric with respect to the centre of the circle circumscribed about this face.

309. Prove that in an equifaced tetrahedron the feet of the altitudes, the midpoints of the altitudes, and the points of intersection of the altitudes of faces lie on the surface of one and the same sphere (12-point sphere).

310. A circle and a point M are given in a plane. The point lies within the circle less than 1/3 of the radius from its centre. Let ABC denote an arbitrary triangle inscribed in a given circle with centre of gravity at the point M. Prove that there are two fixed points in space (D and D') symmetric with respect to the given plane such that the tetrahedrons $ABCD$ and $ABCD'$ are equifaced.

311. A square $ABCD$ is given in a plane. Two points P and Q are taken on the sides BC and CD so that $|CP| + |CQ| = |AB|$. Let M denote a point in space such that in the tetrahedron $APQM$ all the faces are congruent triangles. Determine the locus of projections of points M on the plane perpendicular to the plane of the square and passing through the diagonal AC.

An Orthocentric Tetrahedron

312. In order for the altitudes of a tetrahedron to intersect at one point (such a tetrahedron is called *orthocentric*), it is necessary and sufficient that:

(a) opposite edges of the tetrahedron be mutually perpendicular;

(b) one altitude of the tetrahedron pass through the point of intersection of the altitudes of the base;

(c) the sums of the squares of skew edges be equal;

(d) the line segments connecting the midpoints of skew edges be of equal length;

(e) the products of the cosines of opposite dihedral angles be equal;

(f) the angles between opposite edges be equal.

313. Prove that in an orthocentric tetrahedron the centre of gravity lies at the midpoint of the line segment joining the centre of the circumscribed sphere to the point of intersection of the altitudes.

314. Prove that in an orthocentric tetrahedron the following relationship is fulfilled:

$$|OH|^2 = 4R^2 - 3l^2,$$

where O denotes the centre of the circumscribed sphere, H the point of intersection of the altitudes, R the radius of the circumscribed sphere,

l the distance between the midpoints of the skew edges of the tetrahedron.

315. Prove that in an orthocentric tetrahedron the plane angles adjacent to one vertex are all acute or all obtuse.

316. Prove that in an orthocentric tetrahedron the circles of nine points of each face belong to one sphere (24-point sphere).

317. Prove that in an orthocentric tetrahedron the centres of gravity and the points of intersection of the altitudes of faces, as well as the points dividing the line segments of each altitude of the tetrahedron from the vertex to the point of intersection of the altitudes in the ratio $2:1$, lie on one and the same sphere (12-point sphere).

318. Let H denote the point of intersection of altitudes of an orthocentric tetrahedron, M the centre of gravity of some face, and N one of the points of intersection of the line HM with the sphere circumscribed about the tetrahedron (M lies between H and N). Prove that $|MN| = 2|HM|$.

319. Let G denote the centre of gravity of an orthocentric tetrahedron, F the foot of a certain altitude, K one of the points of intersection of the straight line FG with the sphere circumscribed about the tetrahedron (G lies between K and F). Prove that $|KG| = 3|FG|$.

An Arbitrary Polyhedron. The Sphere

320. Prove that on a sphere it is impossible to arrange three arcs of great circles $300°$ each so that no two have common points.

321. Prove that the shortest line connecting two points on the surface of a sphere is the smaller arc of the great circle passing through these points. (Considered here are lines passing over the surface of the sphere.)

322. Given a polyhedron with equal edges which touch a sphere. Check to see whether there always exists a sphere circumscribed about this polyhedron.

323. Find the area of the triangle formed by the surface of a sphere of radius R intersecting a trihedral angle whose dihedral angles are equal to α, β, and γ, and whose vertex coincides with the centre of the sphere.

324. Let M denote the number of faces, K the number of edges, N the number of vertices of a convex polyhedron. Prove that

$$M - K + N = 2.$$

(Euler was the first to obtain this relationship; it is true not only for convex polyhedra, but also for a broader class of so-called *simply-connected polyhedra.*)

325. Given on the surface of a sphere is a circle. Prove that of all spherical n-gons containing the given circle inside themselves, a regular spherical n-gon has the smallest area.

326. Prove that in any convex polyhedron there is a face having less than six sides.

327. Prove that in any convex polyhedron there is either a triangular face or a vertex at which three edges meet.

328. Prove that a convex polyhedron cannot have seven edges. Prove also that for any $n \geqslant 6$, $n \neq 7$ there is a polyhedron having n edges.

329. Prove that in any convex polyhedron there are two faces with equal number of sides.

330. Found inside a sphere of radius 1 is a convex polyhedron all dihedral angles of which are less than $2\pi/3$. Prove that the sum of the lengths of the edges of this polyhedron is less than 24.

331. The centre of a sphere of radius R is situated outside a dihedral angle of size α at a distance a $(a < R)$ from its edge and lies in the plane of one of its faces. Find the area of the part of a sphere enclosed inside the angle.

332. A ball of radius R touches the edges of a tetrahedral angle each of whose plane angles is equal to 60°. The surface of the ball inside the angle consists of two curvilinear quadrilaterals. Find the areas of these quadrilaterals.

333. Given a cube with edge a. Determine the areas of the parts of the sphere circumscribed about this cube into which it is separated by the planes of the faces of the cube.

334. Given a convex polyhedron. Some of its faces are painted black, no two painted faces having a common edge, and their number being more than half the number of all the faces of the polyhedron. Prove that it is impossible to inscribe a ball in this polyhedron.

335. What is the greatest number of balls with a radius of 7 that can simultaneously touch a ball with a radius of 3 without intersecting one another.

An Outlet into Space

336. Taken on the sides BC and CD of the square $ABCD$ are points M and N so that

$|CM| + |CN| = |AB|$. The lines AM and AN divide the diagonal BD into three segments. Prove that a triangle can always be formed from these segments, one angle of this triangle being equal to 60°.

337. Given in a plane are a triangle ABC and a point P. A straight line l intersects the lines AB, BC, and CA at points C_1, A_1, and B_1, respectively. The lines PC_1, PA_1, and PB_1 intersect the circles circumscribed respectively about the triangles PAB, PBC, and PAC at the respective points C_2, A_2, and B_2, different from the point P. Prove that the points P, A_2, B_2, C_2 lie on one and the same circle.

338. Prove that the diagonals, connecting opposite vertices of the hexagon circumscribed about a circle, intersect at one point (*Brianchon's theorem*).

339. Two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are arranged in a plane so that the lines A_1A_2, B_1B_2, and C_1C_2 intersect at one point. Prove that the three points of intersection of the following three pairs of lines: A_1B_1 and A_2B_2, B_1C_1 and B_2C_2, C_1A_1 and C_2A_2 are collinear (that is, in one straight line) (*Desargues' theorem*).

340. Three planes in space intersect along one straight line. Three trihedral angles are arranged so that their vertices lie on this line, and the edges in the given planes (it is supposed that the corresponding edges, that is, the edges lying in one plane, do not intersect at one point). Prove that the three points of intersection of the corresponding faces of these angles are collinear.

Answers, Hints, Solutions

Section 1

1. $\frac{a^3\sqrt{6}}{108}$. 2. $\frac{4h^3}{45}$. 3. $\frac{5+\sqrt{5}}{24}a^3$. 4. $4a^2$.

5. $\pi-2\arccos\frac{5\pm 2\sqrt{3}}{13}$. 6. $\frac{ab}{2c}$, $\frac{bc}{2a}$, $\frac{ca}{2b}$.

7. $a\sqrt{\frac{1}{3}}$, $\frac{1}{2}$. ?

8. The statement of the problem is obvious for a triangle whose one side lies on the line of intersection of the planes α and β. Then it is possible to prove its validity for an arbitrary triangle, and then also for an arbitrary polygon.

9. Take the triangles AB_1C_1 and AB_2C_2 for the bases of the pyramids $AB_1C_1D_1$ and $AB_2C_2D_2$.

10. The angles under consideration are equal to the angles formed by the diagonal of some rectangular parallelepiped with three edges emanating from its end point.

12. Consider the parallelepiped formed by the planes passing through the edges of the tetrahedron parallel to opposite edges. (This method of completing a tetrahedron to get a parallelepiped will be frequently used in further constructions.) The volume of the tetrahedron is equal to one third the volume of the parallelepiped (the planes of the faces of the tetrahedron cut off the parallelepiped four triangular pyramids, the volume of each of them being equal to one sixth the volume of the parallelepiped), and the volume of the parallelepiped is readily expressed in terms of the given quantities, since the diagonals of its faces are equal and parallel (or, simply, coincide) to the corresponding edges of the tetrahedron, and the altitude of the parallelepiped is equal to the distance between the corresponding edges of the tetrahedron.

13. It is easy to see that each of these relationships (between the areas of the faces and the line segments of

the edge) is equal to the ratio of the volumes of two tetrahedrons into which the given tetrahedron is separated by the bisecting plane.

14. Joining the centres of the sphere to the vertices of the polyhedron, divide it into pyramids whose bases are the faces of the polyhedron, and whose altitudes are equal to the radius of the sphere.

15. It is easy to verify the validity of the given formula for a tetrahedron. Here, two cases must be considered: (1) three vertices of the tetrahedron lie in one plane and one vertex in the other; (2) two vertices of the tetrahedron lie in one plane and two in the other. In the second case, use the formula for the volume of a tetrahedron from Problem 12.

Then note that an arbitrary convex polyhedron can be broken into tetrahedrons whose vertices coincide with those of the polyhedron. This statement is sufficiently obvious, although its proof is rather awkward. Moreover, the suggested formula is also true for nonconvex polyhedra of the indicated type, as well as for solids enclosed between two parallel planes for which the area of the section by a plane parallel to these planes is a quadratic function of the distance to one of them. This formula is named *Simpson's formula*.

16. Since the described frustum of a cone may be considered as the limit of frustums of pyramids circumscribed about the same sphere, for the volume of a frustum of a cone the formula from Problem 14 holds true.

17. First prove the following auxiliary statement. Let the line segment AB rotate about the line l (l does not intersect AB). The perpendicular erected to AB at the midpoint of AB (point C) intersects the line l at point O; MN is the projection of AB on the line l. Then the area of the surface generated by revolving AB about l is equal to $2\pi\,|CO|\cdot|MN|$.

The surface generated by revolving AB represents the lateral surface of the frustum of a cone with radii of the bases BN and AM, altitude $|MN|$, and generatrix AB. Through A draw a straight line parallel to l, and denote by L the point of its intersection with the perpendicular BN dropped from B on l, $|MN| = |AL|$. Denote the projection of C on l by K. Note that the triangles ABL and COK are similar to each other. This taken into consideration, the lateral surface of the frustum of a cone is equal to

$$2\pi \frac{|BN|+|AM|}{2} \cdot |AB| = 2\pi\, |CK| \cdot |AB|$$

$$= 2\pi\, |CO| \cdot |AL| = 2\pi\, |CO| \cdot |MN|.$$

Now, with the aid of the limit passage, it is easy to get the statement of our problem. (If the spherical zone under consideration is obtained by revolving a certain arc $\breve{AB}$ of a circle about its diameter, then the surface area of this zone is equal to the limit of the area of the surface generated by rotating about the same diameter the polygonal line $AL_1L_2 \ldots L_nB$ all vertices of which lie on AB provided that the length of the longest link tends to zero.)

18. Let AB be the chord of the given segment, and O the centre of the circle. Denote by x the distance from O to AB, and by R the radius of the circle. Then the volume of the solid generated by rotating the sector AOB about the diameter will be equal to the product of the area of the surface obtained by revolving the arc $\breve{AB}$ (see Problem 17) by $R/3$, that is, this volume is equal to

$$\frac{1}{3} 2\pi R^2 h = \frac{2}{3} \pi \left(x^2 + \frac{a^2}{4}\right) h = \frac{1}{6} \pi a^2 h + \frac{2}{3} \pi x^2 h.$$

But the second term is equal to the volume of the solid generated by revolving the triangle AOC about the diameter (see the solution of Problem 17). Hence, the first term is just the volume of the solid obtained by revolving the given segment.

19. Place equal loads at the vertices of the pyramid; to find the centre of gravity of the system, you may proceed as follows: first find the centre of gravity of three loads and then, placing a triple load at the found point, find the centre of gravity of the entire system. You may also proceed in a different way: first find the centre of gravity of two loads, then of two others and, finally, the centre of gravity of the whole system. You may not resort to a mechanical interpretation, but, simply, consider the triangle formed by two vertices of the tetrahedron and the midpoint of the opposite edge.

21. Through each edge of the tetrahedron pass a plane parallel to the opposite edge (see the solution of Prob-

lem 12). These planes form a parallelepiped whose edges are equal to the distances between the midpoints of the skew edges of the tetrahedron, and the edges of the tetrahedron themselves are the diagonals of its faces. Then take advantage of the fact that in an arbitrary parallelogram the sum of the squared lengths of the diagonals is equal to the sum of the squared lengths of its sides.

22. If M is the midpoint of BB_1, then A_1M is parallel to CK. Consequently, the desired angle is equal to the angle MA_1D. On the other hand, the plane A_1DM is parallel to CK, hence, the distance between CK and A_1D is equal to the distance from the point K to the plane A_1DM. Denote the desired distance by x, and the dihedral angle by φ. Then we have

$$V_{A_1MDK}=\frac{1}{3}S_{A_1MD}x=\frac{1}{3}S_{A_1KD}a=\frac{a^3}{12}.$$

Hence $x=\dfrac{a^3}{4S_{A_1MD}}$. Find the sides of $\triangle A_1MD$:

$$|A_1D|=a\sqrt{2},\quad |A_1M|=\frac{a\sqrt{5}}{2},\quad |DM|=\frac{3}{2}a.$$

By the theorem of cosines, we find $\cos\varphi=\dfrac{1}{\sqrt{10}}$; thus

$$S_{A_1MD}=\frac{3}{4}a^2,\quad x=\frac{a}{3}.$$

Answer: $\arccos\dfrac{1}{\sqrt{10}},\ \dfrac{a}{3}$.

23. This problem can be solved by the method applied in Problem 22. Here, we suggest another method for determining the distance between skew medians. Let $ABCD$ be the given tetrahedron, K the midpoint of AB, M the midpoint of AC. Project the tetrahedron on the plane passing through AB perpendicular to CK. The tetrahedron is projected into a triangle ABD_1, where D_1 is the projection of D. If M_1 is the projection of M (M_1 is the midpoint of AK), then the distance between the lines CK and DM is equal to the distance from the point K to the line D_1M_1. The distance is readily found, since D_1KM_1 is a right

triangle in which the legs D_1K and KM_1 are respectively equal to $a\sqrt{2/3}$ (altitude of the tetrahedron) and $a/4$.

The problem has two solutions. To get the second solution, consider the medians CK and BN, where N is the midpoint of DC.

Answer: $\arccos\frac{1}{6}$, $a\sqrt{\frac{2}{35}}$ and $\arccos\frac{2}{3}$, $a\frac{\sqrt{10}}{10}$.

24. It follows from the hypothesis that the quadrilateral $ABCD$ is not convex.

Answer: $\frac{\sqrt{3}}{3}$.

25. $\frac{(2b\pm a)a}{2\sqrt{3b^2-a^2}}$. 26. $\frac{41\pi\sqrt{41}}{384}$. 27. $a\sqrt{\frac{7}{8}}$.

28. $a+b\pm\sqrt{2ab-\frac{a^2}{4}}$. 29. $\frac{2a}{3}\sqrt{4R^2-a^2}$.

30. $2+\sqrt{3}$. 31. $\frac{a\sqrt{22}}{8}$.

32. $\frac{3ah}{3a+h(3+2\sqrt{3})}$. 33. $2\arccos\left(\sin\alpha\sin\frac{\pi}{n}\right)$.

34. $12V$. 35. $6R^2-2a^2$. 36. $\frac{\pi}{4}$. 37. $\arctan(2-\sqrt{3})$.

38. If $0<\alpha<\arccos\frac{1}{4}$,

$$l=R\sqrt{27+3\tan^2\frac{\alpha}{2}}\left[\arctan\left(3\cot\frac{\alpha}{2}\right)-\alpha\right];$$

if $\alpha\geqslant\arccos\frac{1}{4}$, $l=0$.

39. $25:20:9$. 40. $\arccos(2-\sqrt{5})$. 41. $\frac{Q^2}{S}$.

42. Denote the side of the base and the altitude of the prism by a, $|KB|=x$. It follows from the hypothesis that the projection of KM on the plane of the base is parallel to the bisector of the angle C of the triangle ABC, that is, $|B_1M|=2x$, $|MC_1|=a-2x$. Let L_1 be the pro-

jection of L on AC. It is also possible to obtain from the hypothesis that $LL_1 = |AL_1| \frac{\sqrt{3}}{2}$, $|L_1C| = a - 2x$. Consequently, the quantity $|AL_1|$ can take on the following values: (1) $|AL_1| = a - |MC_1| = a - (a - 2x) = 2x$; (2) $|AL_1| = a + (a - 2x) = 2(a - x)$. In the first case $|KL|^2 = |KL_1|^2 + |LL_1|^2 = a^2 + 10x^2 - 4ax$; in the second $|KL|^2 = 6(a - x)^2$.

In both cases $|KM|^2 = 3x^2 + a^2$.

Solving two systems of equations, we get two respective values for a:

$$a_1 = \frac{7}{\sqrt{97}},\quad a_2 = \frac{\sqrt{6}+\sqrt{14}}{8}.$$

Answer: $\frac{7}{\sqrt{97}}$, $\frac{\sqrt{6}+\sqrt{14}}{8}$.

43. $\arctan \sqrt{\frac{3}{2}}$.

44. Extend the lateral faces until they intersect. In doing so, we obtain two similar pyramids whose bases are the bases of the given frustum of a pyramid. Let a be the side of the greater base of the frustum, and α the dihedral angle at this base. We can find: the altitude of the greater pyramid $h = a\frac{\sqrt{3}}{6}\tan\alpha$, the radius of the inscribed ball $r = a\frac{\sqrt{3}}{2}\tan\frac{\alpha}{2}$, the altitude of the smaller pyramid $h_1 = h - 2r = a\frac{\sqrt{3}}{6}\left(\tan\alpha - 2\tan\frac{\alpha}{2}\right)$, the side of the smaller base $a_1 = \frac{h_1}{h}a = a\frac{\tan\alpha - 2\tan\frac{\alpha}{2}}{\tan\alpha}$, the lateral edge of the greater pyramid $l = \frac{a\sqrt{3}}{6}\sqrt{\tan^2\alpha + 4}$, the lateral edge of the smaller pyramid $l_1 = l\frac{h_1}{h}$; then take advantage of the condition of existence of a ball touching all the edges of a frustum of a pyramid. This

condition is equivalent to the existence of a circle inscribed in a lateral face, that is, the following equality must be fulfilled:

$$2(l-l_1)=a+a_1.$$

Expressing l, l_1, a_1, in terms of a and α, we get the equation

$$\frac{\sqrt{3}}{3}\sqrt{\tan^2\alpha+4}\cdot\tan\frac{\alpha}{2}=\tan\alpha-\tan\frac{\alpha}{2}.$$

Hence we find $\tan\frac{\alpha}{2}=\sqrt{3}-\sqrt{2}$.

Answer: $2\arctan(\sqrt{3}-\sqrt{2})$.

45. $\frac{-1+\sqrt{5}}{2}<a<\frac{1+\sqrt{5}}{2}$, $a\neq 1$;

$$V=\frac{1}{12}\sqrt{(a^2+1)(3a^2-1-a^4)}.$$

46. $\dfrac{3-\cos\alpha-\cos\beta-\cos\gamma}{3+\cos\alpha+\cos\beta+\cos\gamma}$.

47. If $0<\alpha<\frac{\pi}{6}$, then $S=\frac{3a^2\sqrt{3}}{2\cos\alpha}$; if $\frac{\pi}{6}\leqslant\alpha<\arctan\frac{2}{\sqrt{3}}$, then $S=\frac{a^2}{6\cos\alpha}(18\cot\alpha-3\sqrt{3}-2\sqrt{3}\cot^2\alpha)$; if $\arctan\frac{2}{\sqrt{3}}\leqslant\alpha<\frac{\pi}{2}$, then $S=\frac{a^2}{\sqrt{3}\sin\alpha}(\sqrt{3}+\cot\alpha)$.

48. $\arccos\left(\dfrac{a^2b^2+b^2c^2-c^2a^2}{a^2b^2+b^2c^2+c^2a^2}\right)$.

49. The polyhedron $ABMDCN$ is a triangular prism with base ABM and lateral edges AD, BC, MN.

Answer: $\frac{b}{2a}\sqrt{4a^2-b^2}$.

50. $R=\dfrac{\sqrt{4c^2-a^2b^2}}{2\sqrt{4c^2-a^2-b^2}}$.

51. $\frac{1}{3}\sqrt{3m^2+3n^2+3p^2-a^2-b^2-c^2}$.

52. On the extension of the edge CC_1 take a point K so that B_1K is parallel to BC_1, and through the edge BB_1 pass a plane parallel to the given (Fig. 1). This plane

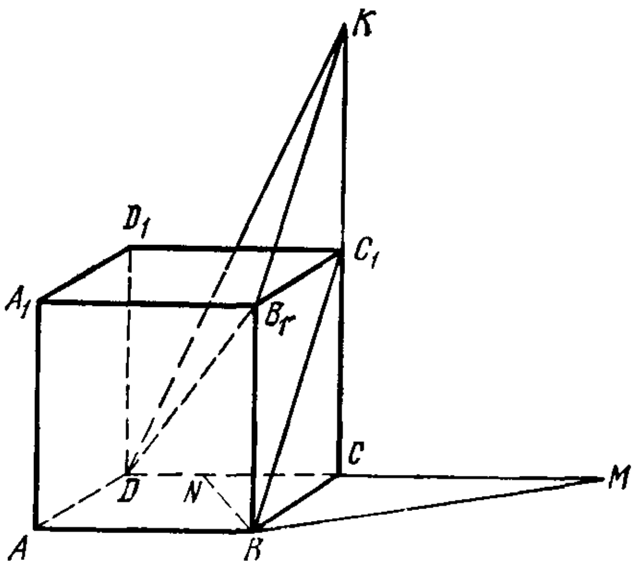


Fig. 1

must pass either through the internal or external bisector of the angle DB_1K. Since the ratio in which the plane passing through BB_1 divides DK_1 is equal to the ratio in which it divides DC, two cases are possible: (1) the plane passes through a point N on the edge DC such that $|DN|/|NC| = \sqrt{3}/\sqrt{2}$, or (2) it passes through a point M on its extension, and once again $|DM|/|MC| = \sqrt{3}/\sqrt{2}$. Find the distance from the point K to the first plane. It is equal to the distance from the point C to the line BN. If this distance is x, then

$$x=\frac{2S_{BNC}}{|BN|}=\frac{a\sqrt{2}}{(\sqrt{3}+\sqrt{2})\sqrt{11-4\sqrt{6}}}$$

$$=\frac{a(\sqrt{6}-1)\sqrt{2}}{5}$$

and

$$\sin\varphi = \frac{x}{|B_1K|} = \frac{\sqrt{6}-1}{5},$$

where φ is the angle between the plane BB_1N and lines B_1D and B_1K. The other angle is found exactly in the same manner.

Answer: $\arcsin\frac{\sqrt{6}\pm 1}{5}$.

53. Let $ABCD$ be the given pyramid whose lateral edges are: $|DA| = a$, $|DB| = x$, $|DC| = y$; by the hypothesis, these edges are mutually perpendicular, and $x + y = a$. It is easy to find that

$$S_{ABC} = \frac{1}{2}\sqrt{a^2(x^2+y^2)+x^2y^2}, \quad V_{ABCD} = \frac{1}{6}axy.$$

On the other hand, if R is the radius of the required ball, then

$$V_{ABCD} = \frac{R}{3}(S_{DAB}+S_{DBC}+S_{DCA}-S_{ABC})$$

$$= \frac{R}{6}\left[ax+ay+xy-\sqrt{a^2(x^2+y^2)+x^2y^2}\right]$$

$$= \frac{R}{6}\left(a^2+xy-\sqrt{a^4-2xya^2+x^2y^2}\right) = \frac{R}{3}xy.$$

Equating the two expressions for V_{ABCD}, we find $R = \frac{a}{2}$.

54. It follows from the hypothesis that the vertex S is projected either into the centre of the circle inscribed in the triangle ABC or into the centre of the circle externally inscribed in it. (An externally inscribed circle touches one side of the triangle and the extensions of two other sides of the triangle.)

Answer: if $\frac{a}{\sqrt{3}} < b \leqslant a$, then $V = \frac{a^2}{12}\sqrt{3b^2-a^2}$; if $a < b \leqslant a\sqrt{3}$, two answers are possible:

$$V_1 = \frac{a^2}{12}\sqrt{3b^2-a^2}, \quad V_2 = \frac{a^2\sqrt{3}}{12}\sqrt{b^2-a^2};$$

if $b > a\sqrt{3}$, three answers are possible:

$$V_1=\frac{a^2}{12}\sqrt{3b^2-a^2},\quad V_2=\frac{a^2\sqrt{3}}{12}\sqrt{b^2-a^2},$$

$$V_3=\frac{a^2\sqrt{3}}{12}\sqrt{b^2-3a^2}.$$

55. Let the angles $\widehat{SAB}$, $\widehat{SCA}$, $\widehat{SAC}$, $\widehat{SBA}$ be equal to $\alpha-2\varphi$, $\alpha-\varphi$, α, $\alpha+\varphi$, respectively. By the theorem of sines, from the triangle SAB we find

$$|SA|=|AB|\frac{\sin(\alpha+\varphi)}{\sin(2\alpha-\varphi)},$$

and from the triangle SAC we find:

$$|SA|=|CA|\frac{\sin(\alpha-\varphi)}{\sin(2\alpha-\varphi)}.$$

But, by the hypothesis, $|AB|=|AC|$. Hence, $\sin(\alpha+\varphi)=\sin(\alpha-\varphi)$, whence $\alpha=\pi/2$. The condition relating the areas of the triangles SAB, ABC, and SAC leads to the equation $\cot^2\varphi\cos 2\varphi=1$, whence $\varphi=\frac{1}{2}\arccos(\sqrt{2}-1)$.

Answer: $\frac{\pi}{2}-\arccos(\sqrt{2}-1)$, $\frac{\pi}{2}-\frac{1}{2}\arccos(\sqrt{2}-1)$, $\frac{\pi}{2}$, $\frac{\pi}{2}+\frac{1}{2}\arccos(\sqrt{2}-1)$.

56. Let $|SA|=l$, l is readily expressed in terms of a, α, and β. If $l\leqslant a$, then $\triangle ASC=\triangle ASB$. (Construct the triangle ASC: take an angle of size α with vertex S, lay off on one side $|SA|=l$, construct a circle of radius a centred at A; since $a\geqslant l$, this circle will intersect the second side of the angle at one point.) And if $l>a$, two cases are then possible: $\triangle ASC=\triangle ASB$ and $\widehat{ACS}=\alpha+\beta$. The line segment l will be less than, equal to, or greater than a according as $2\alpha+\beta$ is greater than, equal to, or less than π.

Besides, in both cases the plane angles adjacent to the vertex A must satisfy the conditions under which a trihedral angle is possible.

Answer. If $\beta > \frac{\pi}{6}$, $2\alpha+\beta \geqslant \pi$, then

$$V = \frac{a^3 \sin(\alpha+\beta)}{12 \sin \alpha} \sqrt{1-2\cos 2\beta};$$

if $\beta \leqslant \frac{\pi}{6}$, $\alpha < \frac{\pi}{3}$, $\alpha+\beta > \frac{\pi}{3}$, then

$$V = \frac{a^3 \sin(\alpha+\beta)}{12 \sin \alpha} \sqrt{3\sin^2\beta - [2\cos(2\alpha+\beta)+\cos\beta]^2};$$

if $\beta > \frac{\pi}{6}$, $\alpha < \frac{\pi}{3}$, $\frac{\pi}{3} < \alpha+\beta < \frac{2\pi}{3}$, then both answers are possible.

57. $\frac{4}{5}$, as measured from the point K.

58. Take C_1 so that $ABCC_1$ is a rectangle (Fig. 2). D_1 is the midpoint of AC_1; O_1, O_2 are the centres of the circles

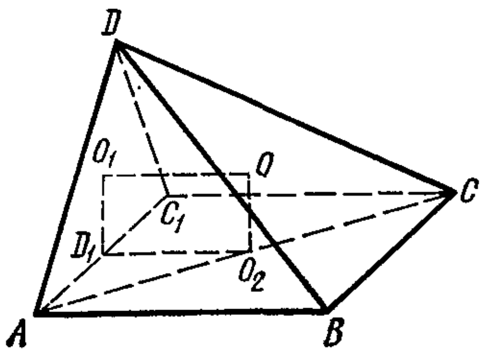


Fig. 2

circumscribed about the triangles AC_1D and ABC, respectively; O is the centre of the sphere circumscribed about $ABCD$. Obviously, O_2 is the midpoint of AC, AB and C_1C are respectively perpendicular to AD and AC_1, consequently, the planes ADC_1 and $ABCC_1$ are mutually perpendicular, and $O_1D_1O_2O$ is a rectangle. Thus $|DC_1| =$

$\sqrt{|DC|^2 - |C_1C|^2} = \sqrt{b^2 - a^2}$, the radius of the circle circumscribed about the triangle DC_1A is

$$R_1 = \frac{|DC_1|}{2\sin \widehat{DAC_1}} = \frac{\sqrt{b^2-a^2}}{2\sin\alpha}.$$

The radius of the sphere $R = |OA|$ can be found from the triangle AO_1O (this triangle is not shown in the figure):

$$R = \sqrt{|AO_1|^2 + |O_1O|^2} = \frac{1}{2}\sqrt{\frac{b^2-a^2}{\sin^2\alpha} + a^2}$$
$$= \frac{1}{2\sin\alpha}\sqrt{b^2 - a^2\cos^2\alpha}.$$

59. Let K be the midpoint of the edge AB of the cube $ABCDA_1B_1C_1D_1$, M the midpoint of the edge D_1C_1, K and M are simultaneously the midpoints of the edges PQ and RS of a regular tetrahedron $PQRS$. D_1C_1 lies on RS. If the edge of the tetrahedron is equal to b, then $|MK| = b\sqrt{2}/2 = a\sqrt{2}$. Hence, $b = 2a$.

Project the tetrahedron on the plane $ABCD$ (Fig. 3): P_1, Q_1, R_1, S_1 are the respective projections of P, Q, R, S

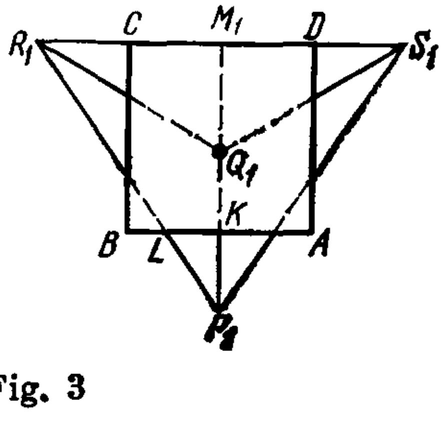


Fig. 3

Since PQ makes an angle of 45° with this plane, the length of P_1Q_1 will be $a\sqrt{2}$.

Let L be the point of intersection of the lines AB and P_1R_1. From the similarity of the triangles P_1LK and $P_1R_1M_1$ we find

$$|LK| = \frac{|R_1M_1|\cdot|P_1K|}{|P_1M_1|} = \frac{a}{1+\sqrt{2}} < \frac{a}{2}.$$

Hence, the edge PR of the tetrahedron (and, consequently, other edges: PS, QR, and QS) pierces the cube.

To compute the volume of the obtained solid, it is convenient to consider the solid as a tetrahedron with corners cut away.

Answer: $\frac{a^3\sqrt{2}}{12}(16\sqrt{2}-17)$.

60. Denote the lengths of these skew edges by a and b, the distance between them by d, and the angle by φ. Using the formula from Problem 15, find the volumes of the obtained parts:

$$V_1=\frac{10}{81}abd\sin\varphi,\quad V_2=\frac{7}{162}abd\sin\varphi.$$

Answer: $\frac{20}{7}$.

61. The area of the projection of the second section on the first plane is half the area of the first section. On the other hand (see Problem 8), the ratio of the area of the projection of the second section to the area of the section itself is equal to $\cos\alpha$.

Answer: $2\cos\alpha$.

62. $\frac{1}{12}\pi R^2H$.

63. If x, y, and z are the respective distances from the centre of the ball to the passed planes, then $x^2+y^2+z^2=d^2$, and the sum of the areas of the three circles will be equal to

$$\pi[(R^2-x^2)+(R^2-y^2)+(R^2-z^2)]=\pi(3R^2-d^2).$$

64. Let $|AC|=x$, $|BD|=y$ (AC and BD touch the ball). D_1 is the projection of D on the plane passing through AC parallel to BD. We have

$$|CD|=x+y=\frac{2R}{\cos\varphi},\quad |CD_1|=2R\tan\varphi.$$

In the triangle CAD_1 the angle CAD_1 is equal either to α or $180°-\alpha$. According to this, x and y must satisfy one of the two systems of equations:

$$\begin{cases} x+y=\dfrac{2R}{\cos\varphi}, \\ x^2+y^2-2xy\cos\alpha=4R^2\tan^2\varphi, \end{cases} \tag{1}$$

or

$$\begin{cases} x+y=\dfrac{2R}{\cos\varphi}, \\ x^2+y^2+2xy\cos\alpha=4R^2\tan^2\varphi. \end{cases} \tag{2}$$

For system (1) we get: $x+y=\dfrac{2R}{\cos\varphi}$, $xy=\dfrac{R^2}{\cos^2\dfrac{\alpha}{2}}$; for system (2): $x+y=\dfrac{2R}{\cos\varphi}$, $xy=\dfrac{R^2}{\sin^2\dfrac{\alpha}{2}}$. Taking into account the inequality $(x+y)^2\geqslant 4xy$, we get that system (1) has a solution for $\varphi\geqslant\dfrac{\alpha}{2}$, and system (2) for $\varphi\geqslant\dfrac{\pi}{2}-\dfrac{\alpha}{2}$. Since the volume of the tetrahedron $ABCD$ is equal to $\dfrac{1}{3}xyR\sin\alpha$, we get the answer: if $\dfrac{\alpha}{2}\leqslant\varphi<\dfrac{\pi}{2}-\dfrac{\alpha}{2}$, the volume of the tetrahedron is equal to $\dfrac{2}{3}R^3\tan\dfrac{\alpha}{2}$; if $\dfrac{\pi}{2}-\dfrac{\alpha}{2}\leqslant\varphi<\dfrac{\pi}{2}$, two values of the volume are possible: $\dfrac{2}{3}R^3\tan\dfrac{\alpha}{2}$ and $\dfrac{2}{3}R^3\cot\dfrac{\alpha}{2}$.

65. Let the common perpendicular to the given edges be divided by the cube into the line segments y, x, and z, $y+x+z=c$ (x is the edge of the cube, y is adjacent to the edge a). The faces of the cube parallel to the given edges cut the tetrahedron in two rectangles, the sides of the first one are equal to $\dfrac{x+z}{c}a$, $\dfrac{yb}{c}$, of the second to $\dfrac{z}{c}a$, $\dfrac{x+y}{c}b$, the smaller sides of these rectangles being equal to the edge of the cube, that is, $\frac{y}{c}b = x$, $\frac{z}{c}a = x$, whence

$$y = \frac{cx}{b}, \quad x = \frac{cx}{a} \text{ and } x = \frac{abc}{ab+bc+ca}.$$

66. Let O_1 and O_2 be projections of the centre of the ball O on the planes KLM and KLN, P the midpoint of ML.

The projections O_1 and O_2 on KL must coincide. It is possible to prove that these projections get into the mid-

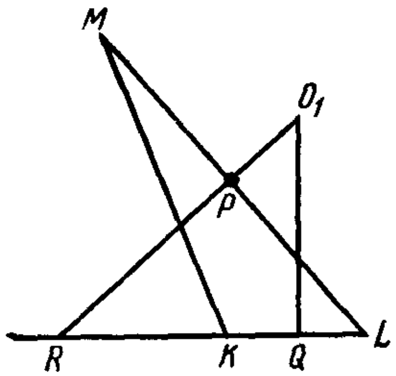


Fig. 4

point of KL, point Q (Fig. 4). Since the dihedral angle between the planes KLM and KLN is equal to 90°, the radius of the desired sphere will be

$$\sqrt{|PO_1|^2 + |O_1Q|^2}.$$

If O_1P is extended to intersect the line KL at point R, then from the right triangle PLR, we find $|RL| = 6a$, $|RP| = 3a\sqrt{3}$. We then find

$$|RQ| = \frac{11a}{2}, \quad |O_1Q| = \frac{11a\sqrt{3}}{6}, \quad |RO_1| = \frac{11a\sqrt{3}}{3},$$

$$|PO_1| = \frac{11a\sqrt{3}}{3} - 3a\sqrt{3} = \frac{2a\sqrt{3}}{3}.$$

Consequently, the radius of the sphere is equal to

$$\sqrt{\frac{4a^2}{3}+\frac{121a^2}{12}}=\frac{a}{2}\sqrt{\frac{137}{3}}.$$

67. Using the equality of tangent lines emanating from one point, prove that the base is a right triangle, and the medians of the lateral faces drawn to the sides of the base are equal. This will imply that the pyramid is regular.

Answer: $\frac{R^3\sqrt{6}}{4}$.

68. The three given angles cannot be adjacent to one face; further, they cannot adjoin to one vertex, since in this case all the line segments joining the midpoints of opposite edges will be equal. It remains only the case when three edges corresponding to right angles form an open polygonal line. Let AB, BC, and CD be the mentioned edges. Denote: $|AB|=x$, $|BC|=y$, $|CD|=z$. Then the distance between the midpoints of AB and CD will be $\sqrt{\frac{x^2}{4}+y^2+\frac{z^2}{4}}$, and between AC and BD (or AD and BC): $\frac{1}{2}\sqrt{x^2+z^2}$. The edge AD will be the greatest: $|AD|=\sqrt{x^2+y^2+z^2}=\sqrt{b^2+3a^2}$.

69. $\pi\frac{4\sqrt{3}-3}{13}$.

70. First prove that $ABCD$ is a rectangle and the plane DEC is perpendicular to the plane $ABCD$. To this end, through E pass a section perpendicular to BC. This section must intersect the base along a straight line passing through M and intersecting the line segments BC and AD (possibly, at their end points). Further, drawing a section which is an isosceles trapezoid through B is only possible if the section contains the edge AB, and $|DE|=|EC|$, $|AE|=|EB|$. Consequently,

$$\frac{3}{5}|AC|\geqslant|ED|=|EC|,\quad \frac{4}{5}|AC|\geqslant|EB|=|AE|,$$

i.e. $|AC|^2 \geqslant |CE|^2 + |AE|^2$ and $\triangle AEC$ is not an acute-angled triangle. But $\widehat{AEC}$ cannot be obtuse, since in that case $\widehat{DEC}$ would also be obtuse.

Thus, $|AC| = \frac{5}{4}|AE| = \frac{5}{3}|EC|$.

Answer: $\frac{3}{8}\sqrt{\frac{65}{14}}$.

71. Through C draw a straight line parallel to AB and take on it a point E such that $|CE| = |AB|$, $ABEC$ is a parallelogram. If O is the centre of the sphere, then the triangle OCE is regular, since $\widehat{OCE} = \pi/3$ and $|CE| = 1$ (it follows from the hypothesis). Hence, the point O is equidistant from all the vertices of the parallelogram $ABEC$. Hence, it follows that $ABEC$ is a rectangle, the projection of O on the plane $ABEC$ is represented by the point K which is the centre of $ABEC$, and $|BD| = 2|OK| = 2\sqrt{|OC|^2 - \frac{1}{4}|BC|^2} = 1$.

72. If x is the area of the sought-for section, $|AB| = a$, then, taking advantage of the formula of Problem 11 for the volume of the pyramid $ABCD$ and its parts, we get

$$\frac{2}{3}\frac{px\sin\frac{\alpha}{2}}{a} + \frac{2}{3}\frac{qx\sin\frac{\alpha}{2}}{a} = \frac{2}{3}\frac{pq\sin\alpha}{a},$$

whence

$$x = \frac{2pq\cos\frac{\alpha}{2}}{p+q}.$$

73. $\frac{8S^2\sin\alpha\sin\beta}{3a\sin(\alpha+\beta)}$.

74. When cutting the ball by the plane AMN, we get a circle inscribed in the triangle AMN. In this triangle $|AN| = a\frac{\sqrt{3}}{2}$, $|AM| = a\frac{\sqrt{3}}{3}$, $|MN| = \frac{a}{2}$ (found from

the triangle CMN). Consequently, if L is the point of contact of the desired ball and AM, then

$$|AL| = \frac{|AN| + |AM| - |MN|}{2} = \left(\frac{5}{12}\sqrt{3} - \frac{1}{4}\right) a.$$

The ball inscribed in $ABCD$ has the radius $r = \frac{1}{4}\sqrt{\frac{2}{3}}\, a$ and touches the plane ACD at point M.

Thus, if x is the radius of the desired ball, then

$$\frac{x}{r} = \frac{|AL|}{|AM|} = \frac{5 - \sqrt{3}}{4}.$$

Hence, $x = \frac{5\sqrt{6} - 3\sqrt{2}}{48}\, a.$

75. $\frac{9\sqrt{3}}{8}$.

76. $\sqrt{3}$.

77. $a\sqrt{2}$.

78. $\arctan \frac{1}{2\sqrt{3}}$.

79. Notation: O is the centre of the sphere; O_1, O_2, O_3 the centres of the given circles, O_4 the centre of the sought-for circle. Obviuosly, the triangle $O_1O_2O_3$ is regular. Find its sides (M is the point of contact of the circles with centres O_1 and O_2). $|O_1M| = |O_2M| = 1$, $|OM| = 2$. Hence, $\widehat{MOO_1} = \widehat{MOO_2} = 30°$, $|OO_1| = |OO_2| = \sqrt{3}$, $|O_1O_2| = \sqrt{3}$. OO_4 is perpendicular to the plane $O_1O_2O_3$ and passes through the centre of the triangle $O_1O_2O_3$, the distances from O_1, O_2, and O_3 to OO_4 are equal to 1. Let K be the point of contact of the circles O_1 and O_4, L the foot of the perpendicular dropped from O_1 on OO_4. KN is perpendicular to LO_1, $|O_1L| = |O_1K| = 1$, $|OO_1| = \sqrt{3}$. From the similarity of the right triangles O_1KN and OO_1L find $|O_1N| = \sqrt{\frac{2}{3}}$. Thus, the required radius $|O_4K| = |LN| = 1 - \sqrt{\frac{2}{3}}$.

80. (a) Since the opposite edges in a regular tetrahedron are perpendicular, the lines C_1E and B_1F must also be perpendicular (Fig. 5).

If K is the midpoint of C_1C, then, since the lines B_1K and B_1A_1 are perpendicular to the line C_1E, the line

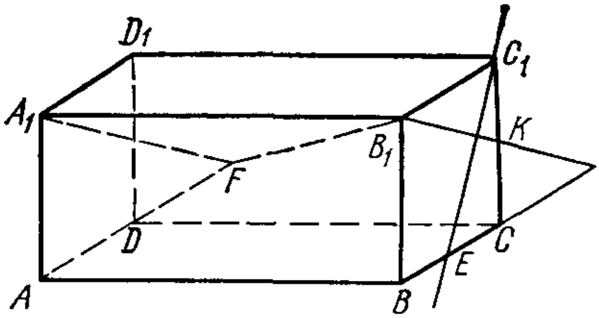


Fig. 5

B_1F must lie in the plane passing through B_1K and B_1A_1, hence it follows that A_1F is parallel to B_1K, and, therefore $|DF| = a$ (this is the answer to this item).

(b) The distance between the midpoints of MN and PQ is equal to the distance between the lines B_1F and C_1E. It can be found by equating different expressions for the volume of the tetrahedron FB_1C_1E:

$$\frac{1}{3} S_{B_1C_1E} 2a = \frac{1}{6} |FB_1| \cdot |C_1E| \cdot x.$$

Hence, $x = \dfrac{4a}{3\sqrt{5}}$.

81. (a) a; (b) $\dfrac{a(2-\sqrt{2})}{2}$.

82. Let $|AB| = a$, then $|AB_1| = |AC_1| = 2.6a$. On the lines AB and AC, take points K and L such that $|AK| = |AL| = |AB_1| = |AC_1| = 2.6a$. An isosceles trapezoid KLC_1B_1 is inscribed in the circle of the base of the cone. All the sides of this trapezoid are readily computed and, hence, the radius of the circle circumscribed about it is also easily found, it equals $\frac{13}{20}\sqrt{7}a$.

It is now possible to find the volume of the cone and prism.

Answer: $\frac{15{,}379\pi}{4800\sqrt{3}}$.

83. Note that the line segment MN is bisected by its point of intersection with the line PQ. Project this line segment on the plane $ABCD$. If N_1 is the projection of N, K_1 the midpoint of AD, Q_1 the midpoint of DC (K_1 and Q_1 are the respective projections of K and Q), then N_1M is perpendicular to AQ_1 and is bisected by the point of intersection. Thus, $\widehat{N_1AD} = 2\widehat{Q_1AD}$. Hence we find $|N_1K_1|$ and then $|N_1M|$.

Answer: $\frac{a}{3}\sqrt{14}$.

84. Through the edge AA_1 pass a plane perpendicular to the plane BCC_1B_1 (Fig. 6). M and N are the points of

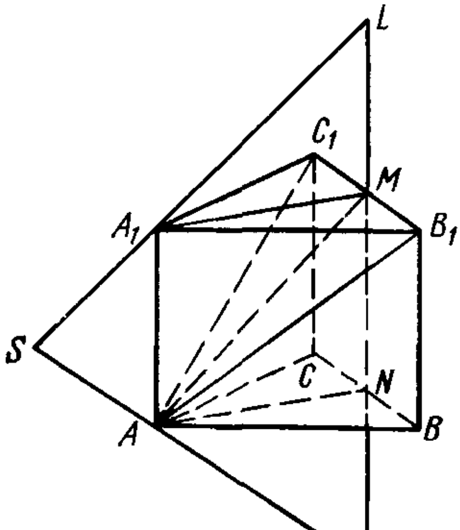


Fig. 6

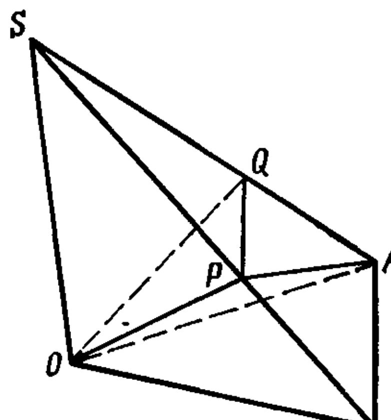


Fig. 7

intersection of this plane with C_1B_1 and CB. Take on MN a point K such that $|NK| = |MN|$. By the hypothesis, AA_1MN is a square, hence, AK is perpendicular to AM, and it follows that AK is perpendicular to the plane AC_1B_1, that is, AK is a straight line along which the planes passing through the vertex A intersect. Analogously, determine the point L for the vertex A_1. The straight lines AK and A_1L intersect at the point S. Thus, our polyhedron represents a quadrangular pyramid $SKPLQ$ with vertex S whose base is found in the plane BB_1C_1C. Further, B_1N is the projection of AB_1. Hence it follows that the plane passing through A perpendicular to AB_1 intersects the plane BB_1C_1C along a straight line perpendicular to B_1N. It follows from the hypothesis that the triangle B_1NC_1 is regular. Hence, the quadrilateral $PLQK$, which is the base of the pyramid $SPLQK$, is a rhombus formed from two regular triangles with side $|KL| = 3a$.

Answer: $\dfrac{9a^3\sqrt{3}}{4}$.

85. The sought-for angle makes the angle between the element OA and the axis of the second cone equal to $\pi/2$. Denote by P and Q the centres of the bases of the given cones, by S the point at which the planes of the bases of the cones intersect the perpendicular erected to the plane OAB at the point O (Fig. 7). In the pyramid $SOAB$: $|OA| = |OB|$, SO is perpendicular to the plane OAB, OP and OQ are respectively perpendicular to SB and SA, $\widehat{POB} = \widehat{QOA} = \varphi$, $\widehat{POQ} = \beta$. Find $\widehat{POA}$. Let $|OA| = |OB| = l$, $|AB| = a$. Then

$$|OP| = |OQ| = l\cos\varphi, \quad |SA| = |SB| = \frac{l}{\sin\varphi},$$

$$|SP| = |SQ| = |OP|\cot\varphi = l\,\frac{\cos^2\varphi}{\sin\varphi},$$

$$|PQ| = |AB|\,\frac{|SP|}{|SB|} = a\cos^2\varphi.$$

On the other hand,

$$|PQ| = 2\,|OP|\sin\frac{\beta}{2} = 2l\cos\varphi\sin\frac{\beta}{2}.$$

Hence

$$a\cos\varphi = 2l\sin\frac{\beta}{2}. \tag{1}$$

Now, find $|PA|$:

$$|PA|^2 = |PB|^2 + |AB|^2 - 2|PB|\cdot|AB|\cos\widehat{PBA}$$
$$= l^2\sin^2\varphi + a^2 - 2l\sin\varphi\cdot a\frac{a\sin\varphi}{2l}$$
$$= l^2\sin^2\varphi + a^2\cos^2\varphi.$$

But if $\gamma = \widehat{POA}$, then from the triangle POA we have: $|PA|^2 = l^2\cos^2\varphi + l^2 - 2l^2\cos\varphi\cos\gamma$.

Equating the two expressions for $|PA|^2$ and taking into consideration (1), find

$$\cos\gamma = \cos\varphi - \frac{2\sin^2\frac{\beta}{2}}{\cos\varphi}.$$

Answer: $\frac{\pi}{2} - \arccos\left(\cos\varphi - \frac{2\sin^2\frac{\beta}{2}}{\cos\varphi}\right)$.

86. $(5\sqrt{6} + \sqrt{22})R$.

87. If the plane cuts the edges AD and CD, then the section represents a triangle and the radius of the inscribed circle will change from 0 to $\frac{a}{\sqrt{2}(2\cos\alpha + \sqrt{4\cos^2\alpha + 1})}$.

Let now the plane cut the edges AB and BC at points P and N, SA and SC at points Q and R, SD at point K, and the extensions of AD and CD at points L and M (Fig. 8). Since the lines PQ and NR are parallel and touch the circle inscribed in our section, PN is the diameter of this circle. Setting $|PN| = 2r$, we have

$$|ML| = 2a\sqrt{2} - 2r,$$
$$|KL| = \frac{a\sqrt{2} - r}{2\cos\alpha}\sqrt{4\cos^2\alpha + 1},$$
$$S_{MKL} = \frac{(a\sqrt{2} - r)^2}{2\cos\alpha}.$$

Thus,

$$r = \frac{a\sqrt{2} - r}{2\cos\alpha + \sqrt{4\cos^2\alpha + 1}},$$

whence

$$r = \frac{a\sqrt{2}}{1 + 2\cos\alpha + \sqrt{4\cos^2\alpha + 1}}.$$

Answer:

$$0 < r \leqslant \frac{a}{\sqrt{2}\,(2\cos\alpha + \sqrt{4\cos^2\alpha + 1})},$$

$$r = \frac{a\sqrt{2}}{1 + 2\cos\alpha + \sqrt{4\cos^2\alpha + 1}}.$$

88. Let us pass a section by the plane passing through the edge AB and the midpoint of CD, point L; K is the

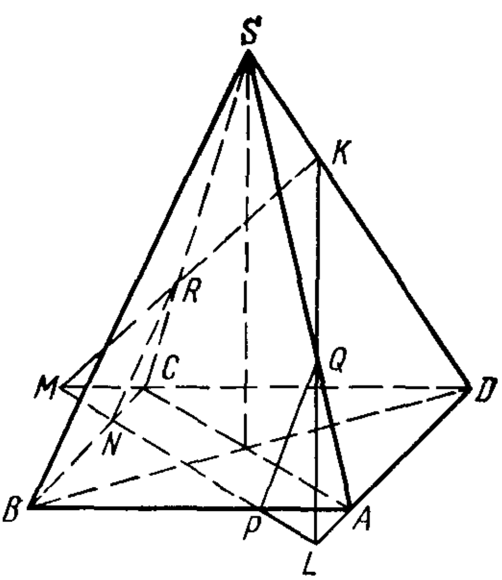


Fig. 8

point of intersection of the plane P and AL. The altitude dropped from A onto BL intersects BK at N and BL at Q (Fig. 9). It is easy to prove that the centre of the sphere lies on the line AQ. Here, the centre of the sphere can lie both on the line segment AN (point O) and on the extension of AQ (point O_1).

The radius of the first sphere is equal to the radius of the circle touching AB and BK and having the centre on

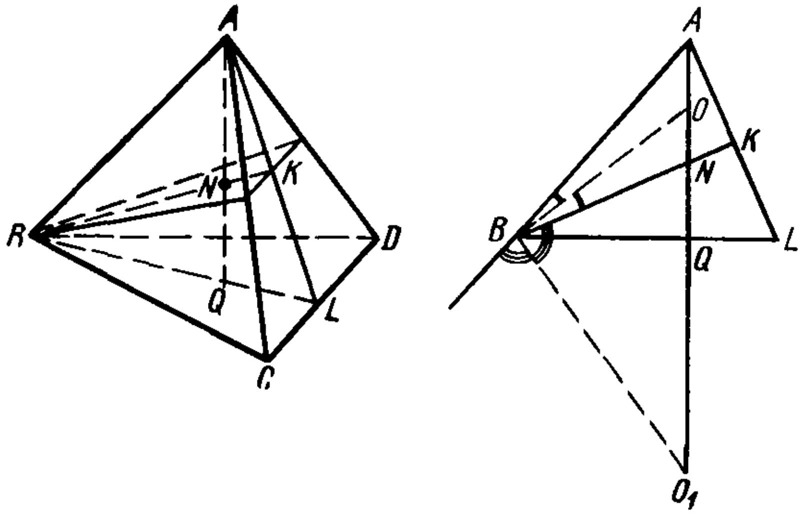


Fig. 9

AN. We denote it by x; x can be found from the relationship

$$S_{BAN}=\frac{1}{2}(|AB|+|BN|)x,$$

$$|BN|=\frac{4}{5}|BK|=\frac{2}{5}\sqrt{2|AB|^2+2|BL|^2-|AL|^2}$$

$$=\frac{\sqrt{11}}{5}a,$$

$$S_{BAN}=\frac{2}{5}S_{BAL}=\frac{\sqrt{2}}{10}a^2,$$

hence, $x=\dfrac{\sqrt{2}\,a}{5+\sqrt{11}}$. The radius of the second sphere is found in the same way.

Answer: $\dfrac{\sqrt{2}\,a}{5\pm\sqrt{11}}$.

89. Let x denote an edge of the tetrahedron, $|MN| = \frac{x}{\sqrt{2}}$. If the edge, whose midpoint is M, makes an angle α with the given plane, then the opposite edge makes an angle of $\frac{\pi}{2} - \alpha$. The projection of the tetrahedron on this plane represents an isosceles trapezoid with bases $x \cos \alpha$ and $x \sin \alpha$ and the distance between the bases equal to $\frac{x}{\sqrt{2}}$. Thus, $S = \frac{x^2}{2\sqrt{2}} (\cos \alpha + \sin \alpha)$. Besides, by the hypothesis, the angle at the greater base is 60°, whence $|\cos \alpha - \sin \alpha| = \sqrt{\frac{2}{3}}$.

Answer: $3S\sqrt{2}$.

90. Let the edge of the cube be equal to 1. Denote by O the centre of the face $ABCD$. From the fact that $\widehat{NMC} = 60°$ and $\widehat{NOC} = 90°$ it follows that O lies between M and C. Setting $|OM| = x$, $|NB| = y$, we have $|MN| = 2x$, $|NO| = x\sqrt{3}$, $|MB| = \sqrt{\frac{1}{2} + x^2}$. Applying the theorem of cosines to the triangles MNB and ONB, we get

$$\begin{cases} \frac{1}{2} + x^2 = 4x^2 + y^2 - 2xy\sqrt{2}, \\ 3x^2 = \frac{1}{2} + y^2 - \frac{2}{\sqrt{3}} y. \end{cases}$$

Hence we find: $x = \frac{1}{\sqrt{6}}$, $y = \frac{2}{\sqrt{3}}$.

Answer: $|AM| : |MC| = 2 - \sqrt{3}$, $|BN| : |ND_1| = 2$.

91. The plane passing through AA_1 parallel to B_1D is parallel to the plane DD_1B_1B. Exactly in the same way, the plane passing through DD_1 parallel to A_1C will be parallel to the plane AA_1C_1C.

On the other hand, the planes passing through the edges BC and B_1C_1 will be parallel to the respective planes AB_1C_1D and A_1BCD_1. This taken into account, construct the section of our polyhedra by the plane parallel to the bases and passing through the midpoints of the lateral edges and the plane passing through the midpoints of the

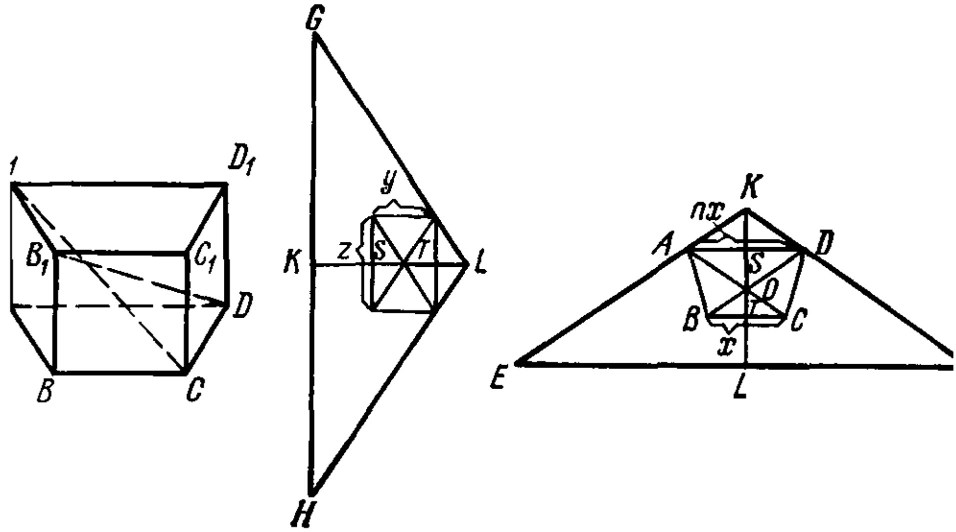


Fig. 10

parallell sides of the bases of the prism (see Fig. 10). In the accompanying figures, L and K are the midpoints of opposite edges EF and HG of the triangular pyramid $EFGH$, the edges EF and HG are mutually perpendicular. Setting $|BC| = x$, $|AD| = nx$, and denoting the altitude of the trapezoid $ABCD$ by y and the altitude of the prism by z, we find

$$|KS| = |SO| = \frac{yn}{n+1}, \quad |TL| = \frac{y}{2},$$

$$|KL| = y\left(\frac{3}{2} + \frac{n}{n+1}\right), \quad |EF| = \frac{5n+3}{2}x,$$

$$|GH| = \frac{5n+3}{n+1}z.$$

The volume of the prism is equal to $\frac{(n+1)\,xyz}{2}$. The volume of the triangular pyramid equals $\frac{1}{6}|EF|\cdot|GH|\times|KL|=\frac{(5n+3)^3}{24\,(n+1)^2}\,xyz.$

Answer: $\frac{(5n+3)^3}{12\,(n+1)^3}$.

92. Let the altitude of the prism be equal to x. On the extension of the edge B_1B take a point K such that $|BK|=\frac{3}{2}x$, $|B_1K|=\frac{5}{2}x$. Since KN is parallel to BM and $|KN|=2|BM|$, the projection of KN on CN is twice the length of the projection of BM on CN, that is, it is equal to $\frac{a}{\sqrt{5}}$. In the triangle CNK, we have $|CN|=\sqrt{a^2+\frac{x^2}{4}}$, $|NK|=\sqrt{a^2+4x^2}$, $|CK|=\sqrt{a^2+\frac{25}{4}x^2}$. Depending on whether the angle C_1NK is acute or obtuse, we shall have two equations

$$a^2+\frac{25}{4}x^2=\left(a^2+\frac{x^2}{4}\right)+(a^2+4x^2)$$

$$-2\sqrt{a^2+\frac{x^2}{4}}\cdot\frac{a}{\sqrt{5}},$$

or

$$a^2+\frac{25}{4}x^2=\left(a^2+\frac{x^2}{4}\right)+(a^2+4x^2)$$

$$+2\sqrt{a^2+\frac{x^2}{4}}\cdot\frac{a}{\sqrt{5}}.$$

Answer: $\frac{a}{2\sqrt{5}}$ or a.

93. Denote two other points of tangency by A_1 and B_1 and the radii of the balls by R and r. In the trapezoid AA_1BB_1 find the bases: $|AA_1|=2R\cos\frac{\alpha}{2}$, $|BB_1|=$

$2r\cos\frac{\alpha}{2}$ and the lateral sides $|AB_1| = |A_1B| = 2\sqrt{Rr}$, and then determine the diagonals $|AB| = |A_1B_1| = 2\sqrt{Rr\left(1+\cos^2\frac{\alpha}{2}\right)}$. If the ball passing through A and A_1 cuts AB at K, then $|A_1B|^2 = |BK|\cdot|BA|$, whence

$$|BK| = \frac{2\sqrt{Rr}}{\sqrt{1+\cos^2\frac{\alpha}{2}}} = \frac{|AB|}{1+\cos^2\frac{\alpha}{2}},$$

$$|AK| = \frac{|AB|\cos^2\frac{\alpha}{2}}{1+\cos^2\frac{\alpha}{2}}.$$

Other parts into which the line segment AB is divided are found in a similar way.

Answer: The line segment AB is divided in the ratio

$$\cos^2\frac{\alpha}{2} : \sin^2\frac{\alpha}{2} : \cos^2\frac{\alpha}{2}.$$

94. It is possible to prove that the axis of the cylinder must pass through the midpoint of the edge BD and belong to the plane BDL, where L is the midpoint of AC. Let the axis of the cylinder make an acute angle α with BD. Projecting the pyramid on a plane perpendicular to the axis of the cylinder, we get a quadrilateral $A_1B_1C_1D_1$ in which $|A_1C_1| = |AC| = 12$. The diagonals A_1C_1 and B_1D_1 are mutually perpendicular, A_1C_1 is bisected by the point F of intersection of the diagonals, and D_1B_1 is divided by F into the line segments $6\sqrt{3}\cos\alpha$ and $10\sqrt{3}\sin\alpha - 6\sqrt{3}\cos\alpha$. From the condition $|A_1F|\cdot|FC_1| = |B_1F|\times|FD_1|$ we get for α the equation

$$\sin^2\alpha - 5\sin\alpha\cos\alpha + 4\cos^2\alpha = 0,$$

whence we find $\tan\alpha_1 = 1$, $\tan\alpha_2 = 4$. But $|B_1D_1| = 10\sqrt{3}\sin\alpha$ and is equal to the diameter of the base of the cylinder. Two values are obtained for the radius of the base of the cylinder: $\frac{5\sqrt{6}}{2}$ and $\frac{20\sqrt{3}}{\sqrt{17}}$.

95. On the edge AS take a point K such that $|AK| = a$. Then the points B, D, and K belong to the section of the cone by a plane parallel to the base of the cone ($|AB| = |AD| = |AK|$). From the fact that C lies in the plane of the base it follows that the plane BDK bisects the altitude of the cone. Thus, the surface area of our cone is four times the surface area of the cone the radius of the base of which is equal to the radius of the circle circumscribed about the triangle BDK with generatrix equal to a.

Answer: $\dfrac{4\pi\sqrt{2}\,a^2(\sqrt{b^2+2a^2}-a)}{\sqrt[4]{b^2+2a^2}\cdot\sqrt{3\sqrt{b^2+2a^2}-4a}}$.

96. Let the radius of the base of the cone be equal to R, altitude to h, the edge of the cube to a. The section of the cone by the plane parallel to the base and passing through the centre of the cube is a circle of radius $R\,\frac{2h-a\sqrt{2}}{2h}$ in which a rectangle (the section of the cube) with sides a and $a\sqrt{2}$ is inscribed, that is,

$$3a^2 = R^2\,\frac{(2h-a\sqrt{2})^2}{h^2}. \tag{1}$$

The section of the cone parallel to the base of the cone and passing through the edge of the cube opposite to the edge lying in the base is a circle of radius $R\,\frac{h-a\sqrt{2}}{h}$. On the other hand, the diameter of this circle is equal to a, that is,

$$a = 2R\,\frac{h-a\sqrt{2}}{h}. \tag{2}$$

From Relationships (1), (2) we get

$$h = \frac{\sqrt{2}\,(5+\sqrt{3})}{4}\,a, \quad R = \frac{2\sqrt{3}-1}{2}\,a.$$

Answer: $\dfrac{\pi(53-7\sqrt{3})\sqrt{2}}{48}$.

97. $\dfrac{3}{5}$.

98. From the equality $\widehat{ACB}=\widehat{ADB}$ and perpendicularity of AB and DC we can obtain that the points C and D are symmetric with respect to the plane passing through AB perpendicular to CD.

Answer: $\dfrac{aS}{3}$.

99. Let K be the midpoint of AB, P the foot of the perpendicular dropped from K on CS. On AB take points M and N such that PMN is a regular triangle (Fig. 11).

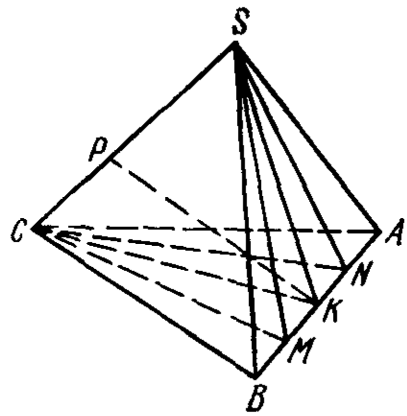


Fig. 11

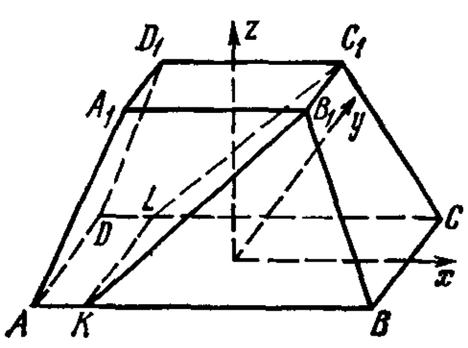


Fig. 12

The pyramid $SPMN$ can be completed to obtain a regular prism $PMNSM_1N_1$ so that PMN and SM_1N_1 will be its bases and PS, MN_1, NN_1 its lateral edges. The prism $A_1B_1CA_2B_2S$ will be homothetic to the prism $PMNSM_1N_1$ with centre in S and ratio of similitude $|CS|/|PS|$. It is easily seen that the sought-for part of the volume of the pyramid $SABC$ contained inside the prism $A_1B_1CA_2B_2S$ is equal to the ratio $|MN|/|AB|$. Setting $AB=a\sqrt{3}$, $|CS|=2a$, we find:

$$|SK|=\frac{\sqrt{13}}{2}a,\quad |CK|=\frac{3}{2}a,\quad |PS|=\frac{5}{4}a,$$

$$|PK| = \frac{3\sqrt{3}}{4}a,$$

$$|MN| = |PK|\frac{2}{\sqrt{3}} = \frac{3}{2}a, \quad |MN|/|AB| = \frac{\sqrt{3}}{2}.$$

Answer: $\frac{\sqrt{3}}{2}$.

100. Let the plane passing through B_1C_1 intersect AB and DC at points K and L (Fig. 12). By the hypothesis, the polyhedra $AKLDA_1B_1C_1D_1$ and $KBCLB_1C_1$ have equal volumes. Apply to them Simpson's formula (Problem 15), setting $|AK| = |DL| = a$. Since the altitudes of these polyhedra are equal, we get the following equation for a:

$$7a+1+4\frac{(a+1)}{2}\cdot\frac{(7+1)}{2} = (7-a)\,7+4\frac{(7-a)}{2}\cdot\frac{(7+1)}{2},$$

whence $a = \frac{16}{5}$.

Denote the altitude of the pyramid by h. Introduce a coordinate system taking its origin at the centre of $ABCD$ and with the x- and y-axes respectively parallel to AB and BC. The points A, C, and D_1 will then have the coordinates $\left(-\frac{7}{2}, -\frac{7}{2}, 0\right)$, $\left(\frac{7}{2}, \frac{7}{2}, 0\right)$, $\left(-\frac{1}{2}, \frac{1}{2}, h\right)$ respectively. It is not difficult to find the equation of the plane ACD_1: $hx - hy + z = 0$. The plane KLC_1B_1 will have the equation $10hx - 8z + 3h = 0$. The normal vector to the former plane is n $(h, -h, 1)$, to the latter m $(10h, 0, -8)$. The condition of their perpendicularity yields $10h^2 - 8 = 0$, $h = \frac{2\sqrt{5}}{5}$. The volume of the pyramid is $\frac{38\sqrt{5}}{5}$.

101. Two cases are possible:

1. The lateral sides of the trapezoid are the projections of the edges AB and B_1C_1. It is possible to prove that in

this case the centre of the sphere is found at the point C. The volume of the pyramid will be equal to $3a^3/8$.

2. The lateral sides of the trapezoid are represented by the projections of the edges AB and A_1C_1. In this case the centre of the sphere is projected into the centre of the circle circumscribed about the trapezoid $ABC_1'A_1'$, the altitude of the trapezoid is equal to $a\sqrt{5}/3$, the volume of the prism is equal to $a^3\sqrt{5}/4$.

Answer: $\frac{3a^3}{8}$ or $\frac{a^3\sqrt{5}}{4}$.

102. $\frac{\pi}{3}a(a^2+2b^2)$.

103. Project the given polyhedra on the plane ABC (Fig. 13). The projections of the points A_1, B_1, and C_1 are not shown in the figure since they have coincided with the

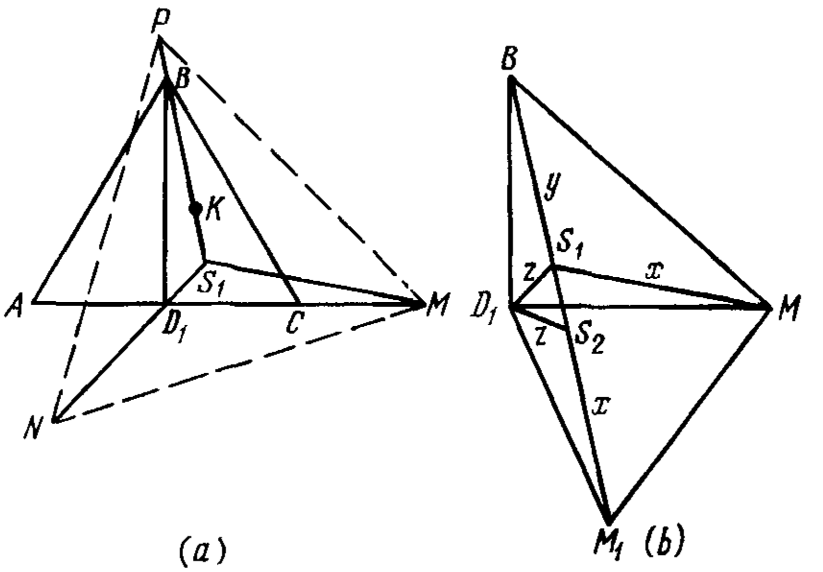


Fig. 13

points A, B, and C; S_1 and D_1 are the respective projections of the points S and D. If on the line segment PS_1 a point K is taken such that $|PK| = |ND_1|$, then the point K is the projection of the point K_1 at which the edge PS intersects the plane $A_1B_1C_1$. Thus, the desired ratio is equal to

$$\frac{|KB|}{|BP|}=\frac{|ND_1|-|PB|}{|PB|}$$
$$=\frac{(|S_1N|-|D_1S_1|)-(|PS_1|-|BS_1|)}{|PS_1|-|BS_1|}$$
$$=\frac{|BS_1|-|D_1S_1|}{|S_1M|-|BS_1|}. \tag{1}$$

Consequently, our problem has reduced to finding the line segments $|S_1M|$, $|BS_1|$, $|D_1S_1|$, where S_1 is a point from which the sides of the triangle BD_1M are seen at equal angles. BD_1M is a right triangle with legs $|D_1M|=2a$, $|BD_1|=a\sqrt{3}$.

Notation: $|S_1M|=x$, $|S_1B|=y$, $|S_1D_1|=z$.

Rotate the triangle D_1S_1M through an angle of $60°$ about the point D_1 (Fig. 13, *b*), $D_1S_1S_2$ is a regular triangle with side z; the points B, S_1, S_2, M_1 are collinear, $\widehat{BD_1M_1}=150°$. From the triangle BD_1M_1 find $x+y+z=a\sqrt{13}$. The altitude of the triangle BD_1M_1 dropped on the side BM_1 is equal to $a\sqrt{\frac{3}{13}}$, whence $z=\frac{2a}{\sqrt{13}}$, $y+\frac{z}{2}=\sqrt{3a^2-\frac{3a^2}{13}}=\frac{6a}{13}$. Now it is easy to find that $y=\frac{5a}{\sqrt{13}}$, $x=\frac{6a}{\sqrt{13}}$. Substituting the found values into (1), we get that the required ratio is equal to 3 (measured from the vertex B).

104. Any tangent plane separates space into two parts; here two cases are possible: either all the three spheres are located in one half-plane or two in one half-plane and one in the other. It is obvious that if a certain plane touches the spheres, then the plane symmetric to it with respect to the plane passing through the centres of the spheres is also tangent to these spheres. Let us show that there is no plane touching the given spheres so that the spheres with radii of 3 and 4 are found on one side of it, while the sphere of radius 6 on the other.

Let the centres of the spheres with radii of 3, 4, and 6 be at the points A, B, and C. The plane touching the given spheres in the above indicated manner divides the sides AC and BC in the ratios $1:2$ and $2:3$, respectively, sthat is, it will pass through points K and L on AC and BC such that $|CK| = 22/3$, $|CL| = 33/5$. The distance from C to KL is easily found, it is equal to $33\sqrt{3/91} < 6$. Hence it follows that through KL it is impossible to pass a plane touching the sphere with radius of 6 and centre at C. We can show that all other tangent planes exist, they will be six all in all.

105. The solution of this problem is based on the fact that the extension of an incident beam is symmetric to the reflected beam with respect to the face from which the beam is reflected. Introduce a coordinate system in a natural way, taking its origin at the point N, and the edges NK, NL, and NM as the x-, y-, and z-axes; denote by Q' and R' the successive points of intersection of the straight line SP with the coordinate planes different from LNM. We have $|PQ| = |PQ'|$, $|QR| = |Q'R'|$.

The point P has the coordinates $(0, 1, \sqrt{3})$. Denote by α, β, α the angles made by the ray SP with the coordinate axes. It follows from the hypothesis that $\beta = \pi/4$, then $\cos\alpha$ is found from the equality $2\cos^2\alpha + \cos^2\beta = 1$, $\cos\alpha = 1/2$ (α is an acute angle). Consequently, the vector $\mathbf{a}$ $(1/2, \sqrt{2}/2, 1/2)$ is parallel to the line SP. If A (x, y, z) is an arbitrary point on this line, then

$$\overrightarrow{OA} = \overrightarrow{OP} + t\mathbf{a},$$

or in coordinate form,

$$x = \frac{t}{2}, \quad y = 1 + \frac{\sqrt{2}}{2}t, \quad z = \sqrt{3} + \frac{t}{2}.$$

The coordinates y and z vanish for $t_1 = -\sqrt{2}$ (this will be point Q') and for $t_2 = -2\sqrt{3}$ (point R'). Thus,

$$Q'\left(-\frac{\sqrt{2}}{2},\ 0,\ \sqrt{3} - \frac{\sqrt{2}}{2}\right), \quad R'(-\sqrt{3},\ 1 - \sqrt{6},\ 0),$$

$$|PQ'| = \sqrt{2}, \quad |Q'R'| = 2\sqrt{3} - \sqrt{2}.$$

Answer: $2\sqrt{3}$.

106. Denote by K the point of tangency of the sphere with the extension of CD, and by M and L the points of tangency with the edges AD and BD, N is the midpoint of BC (Fig. 14). Since $|CD| = |DB| = |DA|$, DN is perpendicular to the plane ABC, $|DK| = |DM| = |DL|$, KL is parallel to DN, ML is parallel to AB, hence, the plane KLM is perpendicular to the plane ABC, $\widehat{KLM} = 90°$. If O is the centre of the sphere, then the line DO is

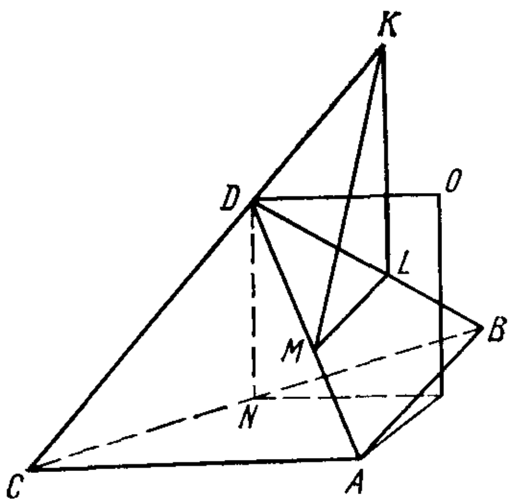


Fig. 14

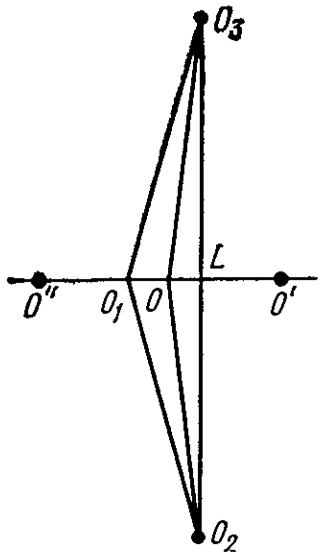


Fig. 15

perpendicular to the plane KLM, that is, DO is parallel to the plane ABC, consequently, $|DN| = 1$ (to the radius of the sphere). In addition, DO passes through the centre of the circle circumscribed about the triangle KLM, that is, through the midpoint of KM. Hence it follows that $\widehat{ODM} = \frac{1}{2}\widehat{KDM}$. Further, $|DA| = |DC| = \sqrt{|CN|^2 + |DN|^2} = \sqrt{3}$, $|CA| = |CB| \cos 30° = \sqrt{6}$, i.e. $\triangle CDA$ is right-angled, $\widehat{CDA} = 90°$, $\widehat{ODM} = 45°$, $|DM| = |OM| = 1$. The required segment of the tangent is equal to $|AM| = |AD| - |DM| = \sqrt{3} - 1$.

107. Let O_1, O_2, O_3 be the points where the balls are tangent to the plane P: O_1 for the ball of radius r, and O_2 and O_3 for the balls of radius R. O is the vertex of the cone (see Fig. 15) and φ the angle between the generatrix of the cone and the plane P. It is possible to show that

$$|O_1O| = r\cot\frac{\varphi}{2}, \quad |OO_2| = |OO_3| = R\cot\frac{\varphi}{2},$$

$$|O_1O_2| = |O_1O_3| = 2\sqrt{Rr}, \quad |O_2O_3| = 2R.$$

Since $|O_1O_2| = |O_1O_3|$, only the angle $O_2O_1O_3$ can be equal to 150°, hence, $R/r = 4\sin^2 75° = 2+\sqrt{3}$.
Further, if L is the midpoint of O_2O_3, then

$$|OL| = \sqrt{|OO_3|^2 - |O_3L|^2} = R\sqrt{\cot^2\frac{\varphi}{2} - 1},$$

$$|O_1L| = \sqrt{|O_1O_3|^2 - |O_3L|^2} = \sqrt{4Rr - R^2}.$$

The point O is found on the line O_1L, and it can lie either on the line segment O_1L itself, or on its extension beyond the points L or O_1 (O' and O'' in the figure). Respectively, we get the following three relationships:

$$|O_1L| = |OO_1| + |OL|, \quad |O_1L| = |O_1O'| - |O'L|,$$

$$|O_1L| = |O''L| - |O''O_1|.$$

Making the substitutions $R = (2+\sqrt{3})\,r$, $\cot\frac{\varphi}{2} = x$ in each of these relationships, we shall come to a contradiction in the first two ($x = 1$ or $x = -2\sqrt{3}/3$), in the third case we find $x = 2\sqrt{3}/3$.

Answer: $\cos\varphi = \frac{1}{7}$.

108. Denote by K and L the midpoints of the edges AD and BC, N and P are the points of intersection of the passed plane and the lines AB and AC, respectively (Fig. 16).

Find the ratios $|PA|/|PC|$ and $|PK|/|PM|$. Draw KQ and AR parallel to DC, Q is the midpoint of AC.

$$|AR| = |DM|, \quad \frac{|PA|}{|PC|} = \frac{|AR|}{|MC|} = \frac{|DM|}{|MC|} = \frac{2}{3},$$

$$\frac{|PK|}{|PM|} = \frac{|KQ|}{|MC|} = \frac{|DC|}{2|MC|} = \frac{5}{6}.$$

Then find

$$\frac{|AN|}{|NB|} = \frac{2}{3}, \quad \frac{|PN|}{|PL|} = \frac{4}{5},$$

$$\frac{V_{PAKN}}{V_{ABCD}} = \frac{|PA|\cdot|AK|\cdot|AN|}{|AC|\cdot|AD|\cdot|AB|} = \frac{2}{5},$$

that is, $V_{PAKN} = 2$. Since the altitude dropped from A on PNK is equal to 1, $S_{PNK} = 6$,

$$\frac{S_{PML}}{S_{PNK}} = \frac{|PK|\cdot|PN|}{|PM|\cdot|PL|} = \frac{3}{2}, \quad S_{PML} = 9.$$

Thus, the area of the section will be $S_{PML} - S_{PNK} = 3$.

109. Knowing the radius of the ball inscribed in the regular triangular pyramid and the altitude of the pyra-

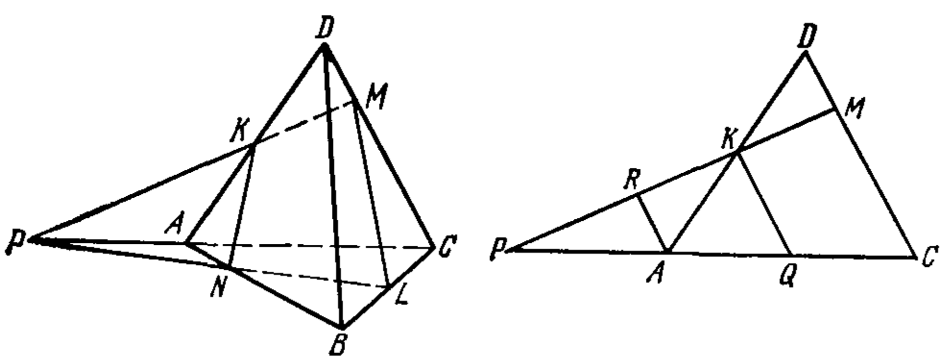


Fig. 16

mid, it is not difficult to find the side of the base. It is equal to 12, $|MK| = |KN|$ (by the hypothesis, the tangents to the ball from the points M and N are equal in length).

Let $|BM| = x$, $|BN| = y$. Finding $|MN|$ by the theorem of cosines from the triangle BMN, and $|MK|$ and $|NK|$ from the respective triangles BMK and BNK, we get the system of equations

$$\begin{cases} x^2+y^2-xy=49, \\ x^2-12x=y^2-12y \end{cases} \Leftrightarrow \begin{cases} x^2+y^2-xy=49, \\ (x-y)(x+y-12)=0. \end{cases}$$

This system has a solution: $x_1 = y_1 = 7$. In this case the distance from K to MN is equal to $4\sqrt{3} - \frac{7\sqrt{3}}{2} = \frac{\sqrt{3}}{2} < 2$, that is, the plane passing through MN and touching the ball actually intersects the extension of SK beyond the point K,

$$|KD| = \frac{12}{13}, \quad |SD| = 6\frac{12}{13}.$$

Another solution of this system satisfies the condition $x + y = 12$. From the first equation we get $(x + y)^2 - 3xy = 49$, $xy = 95/3$. Hence it follows that

$$S_{MKN} = |S_{BMK} + S_{BNK} - S_{BMN}|$$
$$= \left| x\sqrt{3} + y\sqrt{3} - xy\frac{\sqrt{3}}{4} \right| = \frac{49\sqrt{3}}{12}.$$

Consequently, the altitude dropped from K on MN is equal to $\frac{7}{6}\sqrt{3} > 2$, that is, in this case the plane passing through MN and touching the ball does not satisfy the conditions of the problem.

Answer: $6\frac{12}{13}$.

110. From the fact that the edges of the pyramid $ABCD$ touch the ball it follows that the sums of opposite edges of the pyramid are equal. Let us complete the pyramid $ABCD$ to get a parallelepiped by drawing through each edge of the pyramid a plane parallel to the opposite edge.

The edges of the pyramid will be diagonals of the faces of the parallelepiped (Fig. 17), and the edges of the parallelepiped are equal to the distances between the midpoints of the opposite edges of the pyramid. Let $|AD| = a$, $|BC| = b$, then any two opposite edges of the pyramid will be equal to a and b. Let us prove this. Let $|AB| = x$, $|DC| = y$. Then $x + y = a + b$, $x^2 + y^2 = a^2 + b^2$

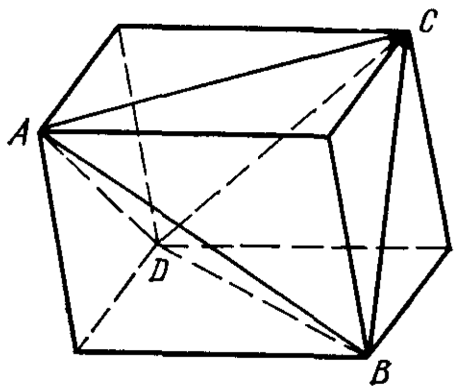


Fig. 17

(the last equality follows from the fact that all the faces of the parallelepiped are rhombi with equal sides).

Hence it follows that $x = a$, $y = b$ or $x = b$, $y = a$. Hence, in the triangle ABC at least two sides are equal in length. But $\widehat{ABC} = 100°$, consequently, $|AB| = x = |BC| = b$, $|AC| = a$, $|DB| = b$, $|DC| = a$.

From the triangle ABC we find $a = 2b \sin 50°$,

$$V_{ABCD} = \frac{1}{3} S_{ADC} h_B = \frac{1}{3} \cdot \frac{a^2 \sqrt{3}}{4} h_B$$

$$= \frac{1}{3} S_{DBC} h_A = \frac{1}{3} \cdot \frac{b^2 \sin 100°}{2} h_A,$$

whence $\dfrac{h_A}{h_B} = \dfrac{a^2 \sqrt{3}}{2b^2 \sin 100°} = \sqrt{3} \tan 50°$.

111. The equality of the products of the lengths of the edges of each face means that the opposite edges of the pyramid are equal in length. Complete the pyramid $SABC$ in a usual way to get a parallelepiped by passing through each edge a plane parallel to the opposite edge. Since the opposite edges of the pyramid $SABC$ are equal in length,

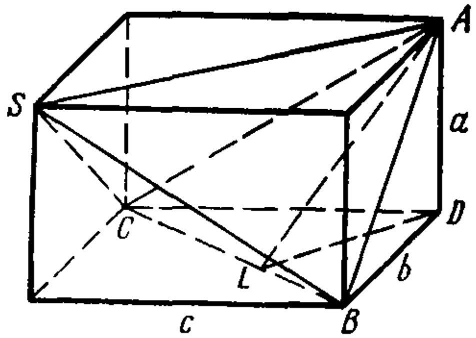


Fig. 18

the obtained parallelepiped will be rectangular. Denote the edges of this parallelepiped by a, b, and c, as is shown in Fig. 18.

In the triangle BCD draw the altitude DL. From the triangle BCD find

$$|DL| = \frac{bc}{\sqrt{b^2+c^2}},$$

$$|AL| = \sqrt{a^2+|DL|^2} = \frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{\sqrt{b^2+c^2}},$$

$$S_{ABC} = \frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}.$$

The volume of the pyramid $SABC$ is one third the volume of the parallelepiped. the altitude on the face ABC is given; thus we get the equation

$$\sqrt{a^2b^2+b^2c^2+c^2a^2}\cdot\sqrt{\frac{102}{55}} = abc. \qquad (1)$$

By the theorem of cosines, for the triangle ABC we get

$$6a^2 = \sqrt{a^2+c^2}\cdot\sqrt{a^2+b^2}\cdot\sqrt{\frac{17}{2}}. \qquad (2)$$

And, finally, the last condition of the problem yields

$$c^2 - 2a^2 - 2b^2 = 30. \tag{3}$$

Solving System (1)-(3), we find $a^2 = 34$, $b^2 = 2$, $c^2 = 102$.

Answer: $\frac{34\sqrt{6}}{3}$.

112. Denote by M and N the points at which the tangents drawn from A and B touch the ball, M_1 and N_1 are projections of the points M and N on the plane ABC (Fig. 19, *a*; the figure represents one of the two equivalent

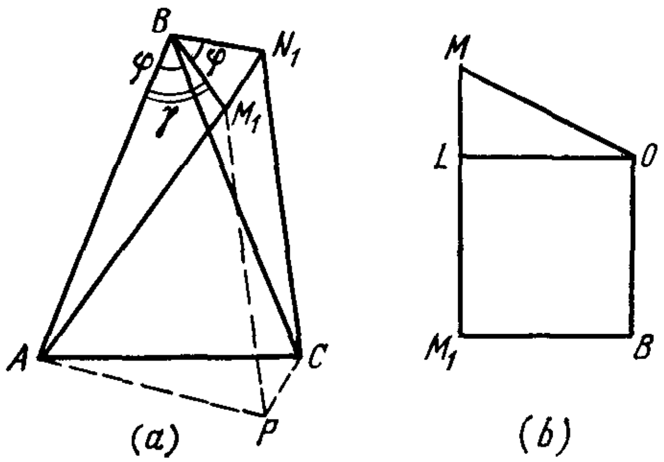


Fig. 19

cases of arrangement of the tangents when these tangents are skew lines; in two other cases these tangents lie in one and the same plane). The following is readily found: $|AM| = |CN| = l$, $|MM_1| = |NN_1| = l \sin\alpha$, $|AM_1| = |CN_1| = l\cos\alpha$. Find $|BM_1|$ and $|BN_1|$ (Fig. 19, *b*; O the centre of the ball, $OL \parallel BM_1$)

$$|BN_1| = |BM_1| = |OL| = \sqrt{r^2 - (l\sin\alpha - r)^2}$$
$$= \sqrt{2rl\sin\alpha - l^2\sin^2\alpha}.$$

When rotated about the point B through an angle $\varphi = \widehat{ABC}$, the point A goes in C, M_1 in N_1, consequently, the triangles BM_1N_1 and BAC are similar,

$$|MN| = |M_1N_1| = |BM_1| \frac{|AC|}{|AB|}$$

$$= \frac{2a}{l} \sqrt{2rl \sin\alpha - l^2 \sin^2\alpha}.$$

Triangle M_1BN_1 is obtained from the triangle ABC by rotating it about B through an angle $\gamma = \widehat{ABM_1}$ followed by a homothetic transformation. Consequently, the angle between M_1N_1 and AC is equal to γ, and since M_1N_1 is parallel to MN, the angle between MN and AC is also equal to γ.

From the triangle BM_1A we find

$$\cos\gamma = \frac{2rl \sin\alpha - l^2 \sin^2\alpha + l^2 - l^2 \cos^2\alpha}{2l \sqrt{2rl \sin\alpha - l^2 \sin^2\alpha}}$$

$$= \frac{r \sin\alpha}{\sqrt{2rl \sin\alpha - l^2 \sin^2\alpha}}.$$

Then

$$\sin\gamma = \frac{\sqrt{2rl \sin\alpha - (l^2 + r^2) \sin^2\alpha}}{\sqrt{2rl \sin\alpha - l^2 \sin^2\alpha}}.$$

Using the obtained values for $|MN|$, $|MM_1|$, and $\sin\gamma$, find the volume of the pyramid $ACMN$:

$$V_{ACMN} = \frac{1}{6} |AC| \cdot |MN| \cdot |MM_1| \sin\gamma$$

$$= \frac{2a^2 \sin\alpha}{3} \sqrt{2rl \sin\alpha - (l^2 + r^2) \sin^2\alpha}. \qquad (1)$$

We now take a point P such that M_1N_1CP is a parallelogram, hence, $MNCP$ is also a parallelogram. Let β be an angle between AM and CN, then $\beta = \widehat{AMP}$. But the triangle ABM_1 is obtained from the triangle CBN_1 by rotating the latter about B clockwise through an angle $\varphi = \widehat{ABC}$. Hence it follows that the angle between AM_1

and CN_1 is equal to φ, and, hence, $\widehat{AM_1P}$ is also equal to φ, that is, the triangles AM_1P and ABC are similar to each other. From this similarity we find $|AP| =$ $2a\cos\alpha$. The angle β is congruent to the angle $\widehat{AMP}$, AMP is an isosceles triangle in which $|AM| = |MP| = l$, $|AP| = 2a\cos\alpha$. Consequently,

$$\sin\frac{\beta}{2}=\frac{a\cos\alpha}{l},$$

$$\sin\beta=2\sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{2a\cos\alpha\sqrt{l^2-a^2\cos^2\alpha}}{l^2}.$$

Express the volume of the pyramid $ACMN$ in a different way:

$$V_{ACMN}=\frac{1}{6}|AM|\cdot|CN|x\sin\beta$$
$$=\frac{1}{3}ax\cos\alpha\sqrt{l^2-a^2\cos^2\alpha},$$

where x is the desired distance. Comparing this formula with the equality (1), we get

$$x=\frac{2a\tan\alpha\sqrt{2rl\sin\alpha-(l^2+r^2)\sin^2\alpha}}{\sqrt{l^2-a^2\cos^2\alpha}}.$$

113. Let $|EA| = x$, the area of the triangle EMA will be the greatest if $|EH| = |HA| = \frac{\sqrt{2}}{2}$, and will equal $\frac{x}{2}\sqrt{\frac{1}{2}-\frac{x^2}{4}}$. The distance from B to the plane EAH is not greater than $|AB| = 1$. Since $S_{AEB} = S_{EBC}$,

$$\frac{1}{12}=\frac{1}{2}V_{ABCEH}=V_{ABEH}\leqslant\frac{x}{12}\sqrt{2-x^2}$$
$$=\frac{1}{12}\sqrt{x^2(2-x^2)}\leqslant\frac{1}{24}[x^2+(2-x^2)]=\frac{1}{12}.$$

Thus, $x = 1$, and the edge AB is perpendicular to the plane EAN; $ABCE$ is a square 1 cm on a side.

Consider two triangular prismatic surfaces: the first is formed by the planes $ABCE$, AHE, and BCH, the second by the planes $ABCE$, ECH, and ABH. Obviously, the radius of the greatest ball contained in the pyramid $ABCEH$ is equal to the radius of the smallest of the balls inscribed in these prisms. And the radius of the ball inscribed in each of these prisms is equal to the radius of the circle inscribed in the perpendicular section. The perpendicular section of the first prism represents a right triangle with legs 1 and 1/2, the radius of the circle inscribed in this triangle is equal to $\frac{3-\sqrt{5}}{4}$. The perpendicular section of the second prism is a triangle AHE, the radius of the circle inscribed in it is equal to $\frac{\sqrt{2}-1}{2} > \frac{3-\sqrt{5}}{4}$.

Answer: $\frac{3-\sqrt{5}}{4}$.

114. From the fact that the straight line perpendicular to the edges AC and BS passes through the midpoint of BS it follows that the faces ACB and ACS are equivalent.

Let $S_{ASB} = S_{BSC} = Q$, then $S_{ACB} = S_{ACS} = 2Q$. Denote by A_1, B_1, C_1, S_1 the projections of M on the respective faces BCS, ACS, ABS, ABC; h_A, h_B, h_C, h_S are the altitudes dropped on these faces, V the volume of the pyramid. Then we shall have

$$|MA_1| + 2|MB_1| + |MC_1| + 2|MS_1| = \frac{3V}{Q}.$$

But, by the hypothesis, $|MB| + |MS| = |MA_1| + |MB_1| + |MC_1| + |MS_1|$. From these two equalities we have:

$$|MB| + |MB_1| + |MS| + |MS_1| = \frac{3V}{Q}.$$

But

$$V = \frac{1}{3} h_S \cdot 2Q = \frac{1}{3} h_B \cdot 2Q = \frac{Q}{3}(h_B + h_S).$$

Consequently, $|MB| + |MB_1| + |MS| + |MS_1| = h_B + h_S$. On the other hand, $|MB| + |MB_1| \geqslant h_B$, $|MS| + |MS_1| \geqslant h_S$. Hence, $|MB| + |MB_1| = h_B$, $|MS| + |MS_1| = h_S$, and the altitudes dropped from B and S intersect at the point M, and the edges AC and BS are mutually perpendicular.

From the conditions of the problem it also follows that the common perpendicular to AC and BS also bisects AC. Let F be the midpoint of AC, and E the midpoint of BS. Setting $|FE| = x$, we get

$$Q = S_{ASB} = \frac{1}{2} |SB| \cdot |AE| = \frac{1}{2} \sqrt{x^2 + \frac{3}{2}},$$

$$2Q = S_{ACB} = \frac{\sqrt{6}}{2} \sqrt{x^2 + \frac{1}{4}}.$$

We shall get the equation $\frac{\sqrt{6}}{2} \sqrt{x^2 + \frac{1}{4}} = \sqrt{x^2 + \frac{3}{2}}$, whence $x = \frac{3}{2}$. Considering the isosceles triangle BFS in which $|BS| = 1$, $|BF| = |FS|$, the altitude $|FE| = \frac{3}{2}$, M the point of intersection of altitudes, we find $|BM| = |SM| = \frac{\sqrt{10}}{6}$.

115. Since the lateral edges of the quadrangular pyramid are equal to one another, its vertex is projected into the point O which is the centre of the rectangle $ABCD$. On the other hand, from the equality of the edges of the triangular pyramid it follows that all the vertices of its base lie on a circle centred at O.

Let the circle on which the vertices of the base of the triangular pyramid lie intersect the sides of the rectangle $ABCD$ at points designated in Fig. 20, *a*. From the fact that the lateral faces of the triangular pyramid are equivalent isosceles triangles it follows that the angles at the vertices of these triangles are either equal or their sum is equal to 180°. Hence, the base is an isosceles triangle. (Prove that it cannot be regular.) Further, two vertices of this triangle cannot lie on smaller sides o the rectangle $ABCD$. If the base will be represented by the triangle LNS, then $|SL| = |LN|$, $\widehat{SLN} = 90°$, and, hence, it will follow that $ABCD$ is a square. But if the triangle LNR will turn out to be the base, then

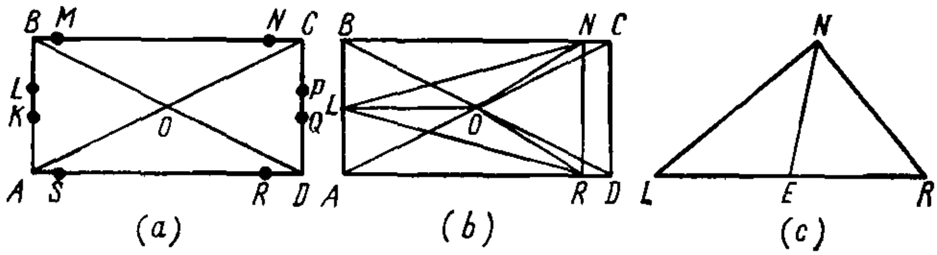


Fig. 20

from the condition $\alpha < 60°$ it will follow that $|BN| > |NR|$. Hence, the sides RL and LN will be equal which is possible when the points K and L coincide with the midpoint of AB.

Reasoning in a similar way, we shall come to another possibility: the vertices of the base of the triangular pyramid are situated at the points R, N, and P, P being the midpoint of CD.

Consider the first case (Fig. 20, *b*). Let $|LO| = |ON| = |OR| = r$. Then $|NR| = |CD| = 2r \tan \frac{\alpha}{2}$.

But, since $\widehat{LEN} + \widehat{NER} = 180°$, the triangles LNE and NER, being brought together (as in Fig. 20, *c*), form a right triangle LNR. Hence,

$$|LN| = \sqrt{4|LE|^2 - |NR|^2}$$

$$= \sqrt{4h^2 + 4r^2 - 4r^2 \tan^2 \frac{\alpha}{2}}.$$

On the other hand,

$$|LN|^2 = \left(r + r\sqrt{1 - \tan^2 \frac{\alpha}{2}}\right)^2 + r^2 \tan^2 \frac{\alpha}{2}.$$

Thus,

$$r^2 = \frac{2h^2}{2\tan^2\frac{\alpha}{2} + \sqrt{1 - \tan^2\frac{\alpha}{2}} - 1}.$$

Considering the triangle NRP in a similar way, we get: $r^2 < 0$.

Answer: $$\frac{8h^3 \tan\frac{\alpha}{2}}{3\left(2\tan^2\frac{\alpha}{2} + \sqrt{1 - \tan^2\frac{\alpha}{2}} - 1\right)}.$$

116. Extend the edge SA beyond the point S, and on the extension take a point A_1 such that $|SA_1| = |SA|$. In SA_1BC the dihedral angles at the edges SA_1 and SC will be equal, and, since $|SA_1| = |SC|$, $|A_1B| = |CB| = b$. The triangle ABA_1 is a right triangle with legs a and b. Consequently, the hypotenuse $|AA_1| = 2|AS| = \sqrt{a^2 + b^2}$.

Answer: $\frac{1}{2}\sqrt{a^2+b^2}$.

117. Consider the tetrahedron with edge $2a$. The surface of the sphere touching all its edges is broken by the surface of the tetrahedron into four equal segments and four congruent curvilinear triangles each of which is congruent to the sought-for triangle. The radius of the sphere is equal to $\frac{a\sqrt{2}}{2}$, the altitude of each segment is equal to $a\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\sqrt{\frac{2}{3}}\right)$, consequently, the area of the sought-for curvilinear triangle is equal to

$$\frac{1}{4}\left[4\pi a^2\left(\frac{\sqrt{2}}{2}\right)^2 - 4\cdot 2\pi a^2\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\sqrt{\frac{2}{3}}\right)\right]$$

$$= \frac{\pi a^2}{6}(2\sqrt{3} - 3).$$

118. Consider the cube with edge equal to $2\sqrt{2}$. The sphere with centre at the centre of the cube touching its edges has the radius 2. The surface of the sphere is broken by the surface of the cube into six spherical segments and eight curvilinear triangles equal to the smallest of the sought-for triangles.

Answer: $\pi(3\sqrt{2}-4)$ and $\pi(9\sqrt{2}-4)$.

119. $\arccos \dfrac{\sqrt{5}-1}{2}$.

120. Pass a section through the axis of the cone. Consider the trapezoid $ABCD$ thus obtained, where A and B are the points of tangency with the surface of one ball, C and D of the other. It is possible to prove that if F is the point of contact of the balls, then F is the centre of the circle inscribed in $ABCD$.

In further problems, when determining the volumes of solids generated by revolving appropriate segments, take advantage of the formula obtained in Problem 18.

121. $\frac{1}{3} SR$.

122. Take advantage of the Leibniz formula (see (1), Problem 153) *

$$3|MG|^2=|MA|^2+|MB|^2+|MC|^2 - \frac{1}{3}(|AB|^2+|BC|^2+|CA|^2),$$

where G is the centre of gravity of the triangle ABC.

If now ABC is the given right triangle, $A_1B_1C_1$ the given regular triangle, G their common centre of gravity, then

$$|A_1A|^2+|A_1B|^2+|A_1C|^2=3|A_1G|^2+\frac{4}{3}b^2 = a^2+\frac{4}{3}b^2.$$

Writing analogous equalities for B_1 and C_1 and adding them together, we obtain that the desired sum of squares is equal to $3a^2+4b^2$.

---

* Here and henceforward (1) means: I.F. Sharygin, *Problems in Plane Geometry* (Nauka, Moscow, 1982).

123. Let the side of the base of the pyramid be equal to a, and the lateral edge to b. Through FE pass a plane parallel to ASC and denote by K and N the points of intersection of this plane with BC and SB. Since E is the midpoint of the slant height of the face SCB, we have $|AF| = |CK| = a/4$, $|SN| = b/4$, $|KE| = 2|EN|$.

Through L draw a straight line parallel to AS and denote its point of intersection with SC by P. We shall have $|SP| = 0.1b$. The triangles LPC and FNK are similar, their corresponding sides are parallel, besides, LM and FE are also parallel, that is, $|PM|/|MC| = |NE|/|EK| = 1/2$, consequently, $|SM| = 0.4b$.

Now, find

$$|LF|^2 - \frac{19}{400}a^2, \quad |ME|^2 = \frac{15}{400}a^2 + \frac{1}{100}b^2.$$

From the condition $|LF| = |ME|$ we get $a = b$. FNK is a regular triangle with side $\frac{3}{4}a$, $|FE|^2 = \frac{7}{16}a^2 = 7$. Consequently, $a = b = 4$.

Answer: $\frac{16}{3}\sqrt{2}$.

124. Prove that the plane cutting the lateral surface of the cylinder divides its volume in the same ratio in which it divides the axis of the cylinder.

Answer: $\frac{\pi a^3}{24}$.

125. Each face of the prism represents a parallelogram. If we connect the point of contact of this face and the inscribed ball with all the vertices of this parallelogram, then our face will be broken into four triangles, the sum of the areas of two of them adjacent to the sides of the bases being equal to the sum of the areas of the other two. The areas of triangles of the first type for all the lateral faces will amount to $2S$. Hence, the lateral area is equal to $4S$, and the total surface area of the prism to $6S$.

126. If the spheres α and β intersected, then the surface area of the part of the sphere β enclosed inside the sphere α would be equal to one fourth the total surface area of the sphere α. (This part would represent

a spherical segment with altitude $\frac{r^2}{2R}$, where r is the radius of α, R the radius of β. Consequently, its surface area will be $2\pi R\frac{r^2}{2R} = \pi r^2$.) Hence, the sphere α contains inside itself the sphere β, and the ratio of the radii is equal to $\sqrt{5}$.

127. When solving this problem, the following facts are used:

(1) the centre of the ball inscribed in the cone lies on the surface of the second ball (consider the corresponding statement from plane geometry);

(2) from the fact that the centre of the inscribed ball lies on the surface of the second ball will follow that the surface area of the inscribed ball will be equal to $4Q$, and its radius will be $\sqrt{Q/\pi}$;

(3) the volume of the frustum of a cone in which the ball is inscribed is also expressed in terms of the total surface area of the frustum and the radius of the ball (the same as the volume of the circumscribed polyhedron), that is, $V = \frac{1}{3}S\sqrt{\frac{Q}{\pi}}$.

128. Prove that if R and r are the radii of the circles of the bases of the frustum of a cone, then the radius of the inscribed ball will be $\sqrt{Rr}$.

Answer: $\frac{S}{2}$.

129. Any of the sections under consideration represents an isosceles triangle whose lateral sides are equal to the generatrix of the cone. Consequently, the greatest area is possessed by the section in which the greatest value is attained by the sine of the vertex angle. If the angle at the vertex of the axial section of the cone is acute, then the axial section has the greatest area. If this angle is obtuse, then the greatest area is possessed by a right triangle.

Answer: $\frac{5}{6}\pi$.

130. Draw SO which is the altitude of the cone to form three pyramids: $SABO$, $SBCO$, and $SCAO$. In each of these pyramids the dihedral angles at the lateral edges SA

and SB, SB and SC, SC and SA are congruent. Denoting these angles by x, y, and z, we get the system

$$\begin{cases} x+y=\beta, \\ y+z=\gamma, \\ z+x=\alpha, \end{cases}$$

whence we find $z=\frac{\alpha-\beta+\gamma}{2}$, and the desired angle will be equal to $\frac{\pi-\alpha+\beta-\gamma}{2}$.

131. The chord BC is parallel to any plane passing through the midpoints of the chords AB and AC. Consequently, the chord BC is parallel to the plane passing through the centre of the sphere and the midpoints of the arcs $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$. Hence it follows that the great circle passing through B and C and the great circle passing through the midpoints of the arcs $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$ intersect at two points K and K_1 so that the diameter KK_1 is parallel to the chord BC.

Answer: $\frac{\pi R}{2} \pm \frac{l}{2}$.

132. It is easy to see that the section of the given solid by a plane perpendicular to the axis of rotation represents an annulus whose area is independent of the distance between the axis of rotation and the plane of the triangle.

Answer: $\frac{\pi a^3 \sqrt{3}}{24}$.

133. If the given plane figure represents a convex polygon, then the solid under consideration consists of a prism of volume $2dS$, half-cylinders with total volume $\pi p d^2$, and a set of spherical sectors whose sum is a ball of volume $\frac{4}{3}\pi d^3$. Consequently, in this case the volume of the solid will be equal to $2dS + \pi p d^2 + \frac{4}{3}\pi d^3$. Obviously, this formula also holds for an arbitrary convex figure

134. Let O be the centre of the ball, CD its diameter, and M the midpoint of BC. Prove that $|AB| = |AC|$. Here, it is sufficient to prove that AM is perpendicular to BC. By the hypothesis, SA is perpendicular to OS, besides, SM is perpendicular to OS (the triangles CSD, CSB, BCD are right triangles, O and M are the respective midpoints of CD and CB). Consequently, the plane AMS is perpendicular to OS, AM is perpendicular to OS. But AM is perpendicular to CD, hence, AM is perpendicular to the plane BCD, thus, AM is perpendicular to BC.

Answer: $\dfrac{Ra^3\sqrt{4b^2-a^2}}{6(4R^2+a^2)}$.

135. In Fig. 21, *a*: $SABC$ is the given pyramid, SO is its altitude, and G is the vertex of the trihedral angle.

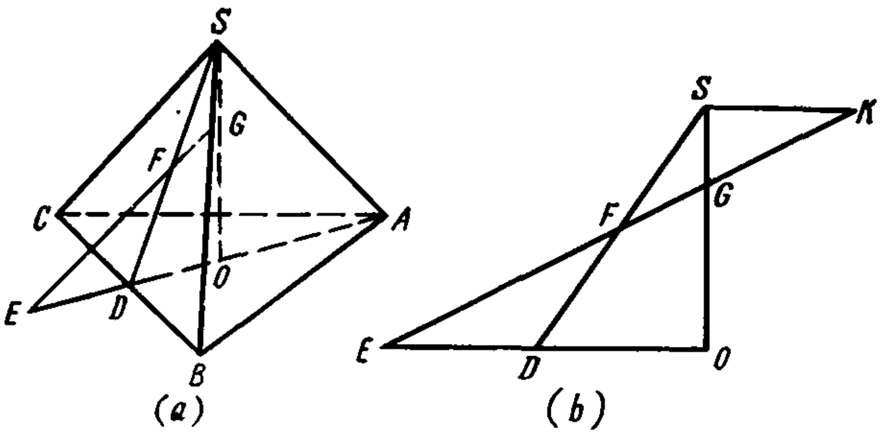


Fig. 21

It follows from the hypothesis, that G lies on SO. Besides, intersecting the plane of the base ABC, the faces of the trihedral angle form a regular triangle whose sides are parallel to the sides of the triangle ABC and pass through its vertices. Consequently, if one of the edges of the trihedral angle intersects the plane ABC at point E and the edge CSB at point F, then F lies on the slant height SD of the lateral face CSB, and $|ED| = |DA|$. By the hypothesis, $|SF| = |FD|$. Through S draw a straight line parallel to EO and denote by K the point of intersection of this line with the line EF (Fig. 21, *b*). We have $|SK| = |ED|$. Hence, $\dfrac{|SG|}{|GO|} = \dfrac{|SK|}{|EO|} = \dfrac{|ED|}{|EO|} = \dfrac{3}{4}$.

Thus, the volume of the pyramid $GABC$ is 4/7 the volume of the pyramid $SABC$.

On the other hand, the constructed trihedral angle divides the portion of the pyramid above the pyramid $GABC$ into two equal parts.

Answer: The volume of the portion of the pyramid outside the trihedral angle is to the volume of the portion inside it as 3 : 11.

136. $\frac{V}{6}$.

137. Figure 22, *a* to *d*, shows the common parts of these two pyramids for all the four cases.

(1) The common part represents a parallelepiped (Fig. 22, *a*). To determine the volume, it is necessary from the volume of the original pyramid to subtract the volumes of three pyramids similar to it with the ratio of similitude 2/3 and to add the volumes of three pyramids also similar to the original pyramid with the ratio of similitude 1/3. Thus, the volume is equal to:

$$V\left[1-3\left(\frac{2}{3}\right)^3+3\left(\frac{1}{3}\right)^3\right]=\frac{2}{9}V.$$

(2) The common part is an octahedron (Fig. 22, *b*) whose volume is

$$V\left[1-4\left(\frac{1}{2}\right)^3\right]=\frac{V}{2}.$$

(3) The common part is represented in Fig. 22, *c*. To determine its volume it is necessary from the volume of the original pyramid to subtract the volume of the pyramid similar to it with the ratio of similitude equal to 1/3 (in the figure this pyramid is at the top), then to subtract the volumes of three pyramids also similar to the original pyramid with the ratio of similitude equal to 5/9 and to add the volumes of three pyramids with the ratio of similitude equal to 1/9. Thus, the volume of the common part is equal to

$$V\left[1-\left(\frac{1}{3}\right)^3-3\left(\frac{5}{9}\right)^3+3\left(\frac{1}{9}\right)^3\right]=\frac{110}{243}V.$$

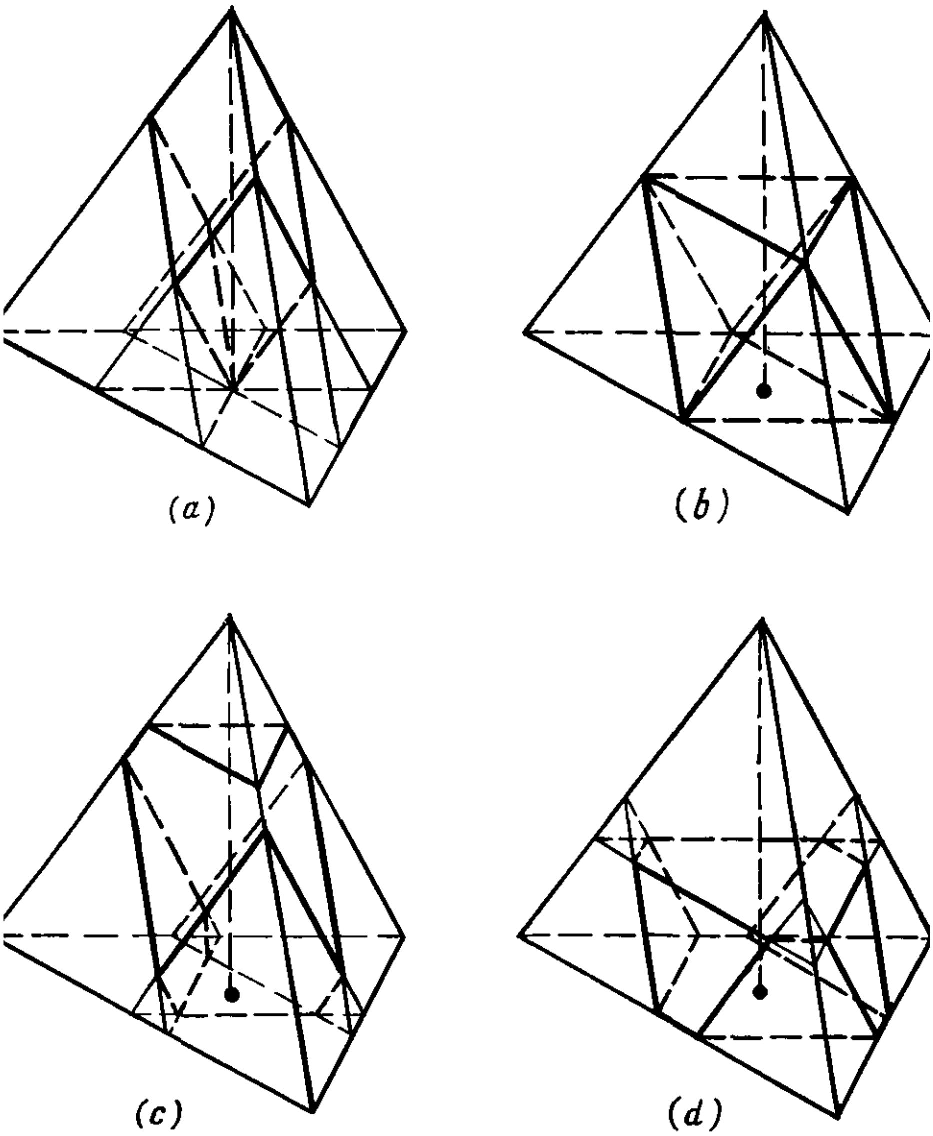


Fig. 22

(4) The common part is represented in Fig. 22, *d*. Its volume is

$$V\left[1-\left(\frac{3}{5}\right)^3-3\left(\frac{7}{15}\right)^3+3\left(\frac{1}{15}\right)^3\right]=\frac{12}{25}V.$$

138. Let the edge of the regular tetrahedron $ABCD$ be equal to a, and K and L be the midpoints of the edges

CD and AB (Fig. 23). On the edge CB take a point M and through this point draw a section perpendicular to KL. Setting $|CM| = x$, determine the quantity x for which the rectangle obtained in our section will have the angle

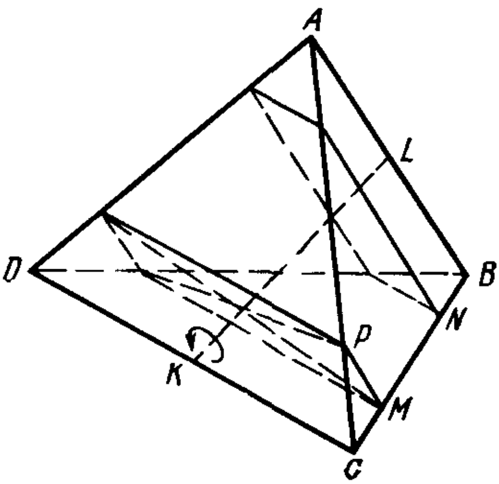


Fig. 23

between the diagonals equal to α. Since the sides of the obtained rectangle are equal to x and $a - x$, x can be evaluated from the following equation:

$$\frac{x}{a-x} = \tan\frac{\alpha}{2}, \quad x = \frac{a \tan\frac{\alpha}{2}}{1+\tan\frac{\alpha}{2}}.$$

If we take on the edge BC one more point N such that $|BN| = |CM| = x$, and through this point draw a section perpendicular to KL, then we shall obtain another rectangle with the angle between the diagonals equal to α. Hence it follows that, on being rotated anticlockwise about KL through an angle α, the plane BCD will pass through the points K, P, and N. Thus, on being rotated, the plane BCD will cut off the tetrahedron $ABCD$ a pyramid $KPNC$ whose volume is equal to

$$\frac{|KC|}{|CD|}\cdot\frac{|CP|}{|CA|}\cdot\frac{|CN|}{|CB|}V_{ABCD}=\frac{x(a-x)}{2a^2}V$$

$$=\frac{\tan\frac{\alpha}{2}}{2\left(1+\tan\frac{\alpha}{2}\right)^2}V.$$

Similar reasoning will do for any face of the tetrahedron. Consequently, the volume of the common part will be equal to $\dfrac{1+\tan^2\frac{\alpha}{2}}{\left(1+\tan\frac{\alpha}{2}\right)^2}V.$

139. Let the cube $ABCDA_1B_1C_1D_1$ be rotated through an angle α about the diagonal AC_1 (Fig. 24). On the edges

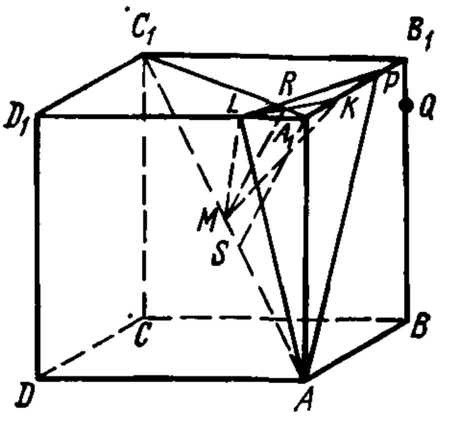


Fig. 24

A_1B_1 and A_1D_1 take points K and L such that $|A_1K| = |A_1L| = x$, from K and L drop perpendiculars on the diagonal AC_1; since the cube is symmetric with respect to the plane ACC_1A_1, these perpendiculars will pass through one point M on the diagonal AC_1. Let x be chosen so that $\widehat{KML} = \alpha$. Then, after rotating about the diagonal AC_1 anticlockwise (when viewed in the direction from A to C_1) through an angle α, the point K will move into L. On the edges B_1A_1 and B_1B take points P and Q at the same distance x from the vertex B_1. After the same rotation the point Q will move into P. Consequently, after the rotation the face ABB_1A_1 will pass through the points A, L, and P and will cut off our cube a pyramid AA_1PL whose volume is equal to $\frac{1}{6}ax(a-x)$. The same reasoning is true for all the faces. Thus, the volume of the common part is equal to $a^3 - ax(a-x)$. It now remains to find x from the condition $\widehat{KML} = \alpha$. To this end, join M to the midpoint of the line segment LK, point R. We have

$$|MR| = x\frac{\sqrt{2}}{2}\cot\frac{\alpha}{2}, \quad |C_1R| = a\sqrt{2} - x\frac{\sqrt{2}}{2},$$

and from the similarity of the triangles C_1RM and C_1A_1A find $x = \dfrac{2a}{1+\sqrt{3}\cot\frac{\alpha}{2}}$.

Thus, the volume of the common part is equal to

$$\frac{3a^3\left(1+\cot^2\frac{\alpha}{2}\right)}{\left(1+\sqrt{3}\cot\frac{\alpha}{2}\right)^2}.$$

140. Let A be some point on the ray, B the point of incidence of the ray on the mirror, K and L the projections of A on the given mirror and rotated mirror, A_1 and A_2 the points symmetric to A with respect to these mirrors, respectively. The sought-for angle is equal to the angle $\widehat{A_1BA_2}$. If $|AB| = a$, then $|A_1B| = |A_2B| = a$, $|AK| = a\sin\alpha$. Since $\widehat{KAL} = \beta$, we have $|KL| = |AK|\sin\beta = a\sin\alpha\sin\beta$, $|A_1A_2| = 2|KL| = 2a\sin\alpha\sin\beta$. Thus, if φ is the desired angle, then $\sin\frac{\varphi}{2} = \sin\alpha\sin\beta$.

Answer: $2\arcsin(\sin\alpha\sin\beta)$.

141. Fix the triangle ABC, then known in the triangle ADC are two sides $|AC|$ and $|DC|$ and the angle $\widehat{ADC} = \alpha$. In the plane of the triangle ADC construct a circle of radius $|AC|$ centred at C (Fig. 25, *a*). If $\alpha \geqslant 60°$,

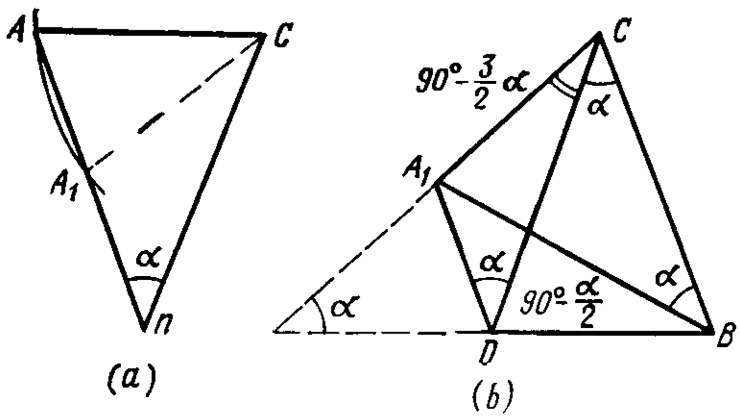


Fig. 25

then there exists only one triangle having the given sides and angle (the second point A_1 will turn out to lie on the other side of the point D); this is a triangle congruent to the triangle ABC. In this case AC and BD are mutually perpendicular.

And if $\alpha < 60°$, then there is another possibility (in Fig. 25, *a*, this is the triangle A_1DC). In this triangle $\widehat{CA_1D} = 90° + \frac{\alpha}{2}$, $\widehat{A_1CD} = 90° - \frac{3\alpha}{2}$. But in this case the vertex C (Fig. 25, *b*) is common for the angles $\widehat{BCA_1} = 90° - \frac{\alpha}{2}$, $\widehat{BCD} = \alpha$, $\widehat{A_1CD} = 90° - \frac{3\alpha}{2}$, and since $90° - \frac{\alpha}{2} = \left(90° - \frac{3\alpha}{2}\right) + \alpha$, the points A_1, B, C, and D lie in the same plane, and the angle between A_1C and BD will be equal to α.

Answer: if $\alpha \geqslant 60°$, then the angle between AC and BD is equal to $90°$, if $\alpha < 60°$, then the angle between AC and BD can be equal to either $90°$ or α.

142. Let the base of the prism be the polygon $A_1A_2 \ldots A_n$, O the centre of the circle circumscribed about it. Let then a certain plane cut the edges of the prism at points $B_1, B_2, \ldots, B_n$, and M be a point in the plane such that the line MO is perpendicular to the plane of the base of the prism. Then the following equalities hold:

$$\sum_{k=1}^{n} |A_kB_k| = n\,|MO|, \tag{1}$$

$$V = S\,|MO|, \tag{2}$$

where V is the volume of the part of the prism enclosed between the base and the passed plane.

Prove Equality (1). For an even n it is obvious. Let n be odd. Consider the triangle $A_kA_{k+1}A_l$, where A_l is the vertex most distant from A_k and A_{k+1}. Let C_k and C'_k be the midpoints of A_kA_{k+1} and B_kB_{k+1}, respectively. Then $\frac{|C_kO|}{|OA_l|} = \cos\frac{\pi}{n} = \lambda$. Now, it is easy to prove that

$$|MO| = \frac{|C_kC'_k| + |A_lB_l|\lambda}{1+\lambda}$$

$$= \frac{\frac{1}{2}(|A_kB_k| + |A_{k+1}B_{k+1}|) + |A_lB_l|\,\lambda}{1+\lambda}.$$

Adding these equalities for all k's (for $k = n$ instead of $n+1$ take 1), we get Statement (1).

To prove Equality (2), consider the polyhedron $A_kA_{k+1}OB_kB_{k+1}M$. If now V_k is the volume of this polyhedron, then, by Simpson's formula, we have (see Problem 15)

$$V_k = \frac{b_n}{6}\left(\frac{|A_kB_k| + |A_{k+1}B_{k+1}|}{2}\,a_n + 4\,\frac{|A_kB_k| + |A_{k+1}B_{k+1}| + 2\,|MO|}{4}\cdot\frac{a_n}{2}\right)$$

$$= a_nb_n\,(|A_kB_k| + |A_{k+1}B_{k+1}| + |MO|)$$

$$= \frac{S}{3n}\,(|A_kB_k| + |A_{k+1}B_{k+1}| + |MO|),$$

where a_n, b_n are the side and the slant height of the polygon $A_1A_2 \ldots A_n$. Adding these equalities for all K's and taking (1) into consideration, we get Equality (2).

Now, it is not difficult to conclude that the answer to our problem will be the quantity $\frac{nV}{S}$.

143. Let the pentagon $ABCDE$ be the projection of the regular pentagon, where $|AB| = 1$, $|BC| = 2$, $|CD| = a$, $ABCD$ is a trapezoid in which $\frac{|AD|}{|BC|} = \lambda = \frac{1+\sqrt{5}}{2}$, F the point of intersection of its diagonals, $AFDE$ is

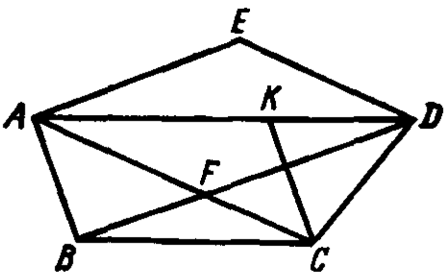


Fig. 26

a parallelogram. Draw CK parallel to AB (Fig. 26). In the triangle CKD we have: $|CK| = 1$, $|KD| = 2(\lambda - 1)$, $|CD| = a$. Set $\widehat{CDK} = \varphi$. Write the theorem of cosines for the triangles CKD and ACD:

$$1 = a^2 + 4(\lambda - 1)^2 - 4(\lambda - 1)\, a \cos \varphi,$$

$$|AC|^2 = a^2 + 4\lambda^2 - 4a\lambda \cos \varphi.$$

From these two relationships we find

$$|AC| = \sqrt{\frac{4\lambda^2 - 3\lambda - a^2}{\lambda - 1}},$$

$$|ED| = |AF| = \frac{\lambda}{\lambda + 1}\sqrt{\frac{4\lambda^2 - 3\lambda - a^2}{\lambda - 1}}.$$

Similarly, we find

$$|AE| = |FD| = \frac{\lambda}{\lambda+1}\sqrt{\frac{a^2\lambda-1+4\lambda^2-4\lambda}{\lambda-1}}.$$

Answer: Two other sides are equal to

$$\frac{\sqrt{5}-1}{4}\sqrt{14+10\sqrt{5}-2(\sqrt{5}+1)a^2}$$

and

$$\frac{\sqrt{5}-1}{4}\sqrt{a^2(6+2\sqrt{5})+6(\sqrt{5}+1)}.$$

The problem has a solution for $\sqrt{5}-2 < a < \sqrt{5}$.

144. Let the edge of the cube be equal to a, $|NC_1| = x$. Find

$$|LM| = \frac{a}{2}, \quad |NK| = \frac{x}{\sqrt{2}},$$

$$|LN|^2 = |LB_1|^2 + |B_1N|^2 = \frac{a^2}{4} + (a-x)^2$$

$$= \frac{5}{4}a^2 - 2ax + x^2,$$

$$|LK|^2 = |LB_1|^2 + |B_1K|^2$$

$$= |LB_1|^2 + |B_1N|^2 + |NK|^2$$

$$+2|B_1N|\cdot|NK|\frac{\sqrt{2}}{2}$$

$$= \frac{a^2}{4} + (a-x)^2 + \frac{x^2}{2} + (a-x)x$$

$$= \frac{5}{4}a^2 - ax + \frac{x^2}{2},$$

$$|MN|^2 = |MB_1|^2 + |B_1N|^2 = \frac{3a^2}{2} - 2ax + x^2,$$

$$|MK|^2 = |MB|^2 + |BK|^2 - |MB|\cdot|BK|$$

$$= \frac{3a^2}{2} - \frac{3}{2}ax + \frac{x^2}{2}.$$

If $\widehat{LMK} = \widehat{MKN} = \varphi$, then by the theorem of cosines, for the triangles LMK and MKN we get:

$$|LK|^2 = |LM|^2 + |MK|^2 - 2|LM|\cdot|MK|\cos\varphi,$$
$$|MN|^2 = |MK|^2 + |KN|^2 - 2|MK|\cdot|KN|\cos\varphi.$$

Eliminating $\cos\varphi$ from these equations, we get

$$|LK|^2\cdot|KN| - |MN|^2\cdot|LM|$$
$$=(|LM| - |KN|)(|LM|\cdot|KN| - |MK|^2).$$

Expressing the line segments entering this equality with the aid of the found formulas, we get

$$\left(\frac{5a^2}{4} - ax + \frac{x^2}{2}\right)\frac{x}{\sqrt{2}} - \left(\frac{3a^2}{2} - 2ax + x^2\right)\frac{a}{2}$$
$$=\left(\frac{a}{2} - \frac{x}{\sqrt{2}}\right)\left(\frac{ax}{2\sqrt{2}} - \frac{3a^2}{2} + \frac{3ax}{2} - \frac{x^2}{2}\right).$$

From this equation we find $x = a\left(1 - \frac{\sqrt{2}}{2}\right)$.

Answer: $\frac{|B_1N|}{|NC_1|} = \sqrt{2} + 1$.

145. Two cases are possible: (1) the centre of the circumscribed sphere coincides with the centre of the base and (2) the centre of the circumscribed sphere is found at the point of the surface of the inscribed sphere diametrically opposite to the centre of the base.

In the second case, denoting by R and r the radii of the respective inscribed and circumscribed spheres, find the altitude of the pyramid $2r + R$ and the side of the base $\sqrt{R^2 - 4r^2}$. The section passing through the altitude and midpoint of the side of the base is an isosceles triangle with altitude $R + 2r$, base $\sqrt{3(R^2 - 4r^2)}$ and radius of the inscribed circle equal to r. Proceeding from this, it is possible to get the relationship $3R^2 - 6Rr - 4r^2 = 0$ for R and r.

Answer: $\frac{3+\sqrt{21}}{3}$ (in both cases).

146. Two cases are possible: (1) the centre of the circumscribed ball coincides with the centre of the base, (2) the centre of the circumscribed sphere is found at the point of the surface of the inscribed ball diametrically opposite to the centre of the base. In the first case, the plane angle at the vertex is equal to $\pi/2$.

Consider the second case. Denote by a, b, and l the side of the base, lateral edge, and the slant height of the lateral face, respectively. Then

$$b^2=l^2+\frac{a^2}{4}, \tag{1}$$

the radius r of the inscribed ball is equal to the radius of the circle inscribed in the isosceles triangle with base a and lateral side l:

$$r=\frac{a\sqrt{2l-a}}{2\sqrt{2l+a}}, \tag{2}$$

the radius R of the circumscribed ball is equal to the radius of the circle circumscribed about the isosceles triangle with base $a\sqrt{2}$ and lateral side b:

$$R=\frac{b^2\sqrt{2}}{2\sqrt{2b^2-a^2}}. \tag{3}$$

Here, the centre of the circle must lie inside the triangle, which means that $b > a$. Since the distance from the centre of the circumscribed ball to the base is $2r$, we have $R^2-\frac{a^2}{2}=4r^2$. Substituting the values of R and r expressed by Formulas (2) and (3) into this equality, we get after simplification:

$$\frac{(b^2-a^2)^2}{2(2b^2-a^2)}=\frac{a^2(2l-a)}{2l+a}.$$

Expressing b in terms of a and l by Formula (1), we get

$$\left(l^2-\frac{3a^2}{4}\right)^2=a^2(2l-a)^2.$$

Taking into account that $b \geq a$ or $l > a\,\frac{\sqrt{3}}{2}$, we obtain that a and l satisfy the equation

$$l^2 - \frac{3a^2}{4} = a\,(2l - a),$$

whence $\frac{l}{a} = 1 + \frac{\sqrt{3}}{2}$ $\left(\text{for the second root } \frac{l}{a} < \frac{\sqrt{3}}{2}\right)$.

Answer: $\frac{\pi}{2}$ or $\frac{\pi}{6}$.

147. Let K be the projection of the vertex S on the plane $ABCD$, and let L, M, N, and P be the projection of S on the respective sides AB, BC, CD, and DA.

It follows from the hypothesis that LSN and MSP are right triangles with right angles at the vertex S. Consequently, $|LK|\cdot|KN| = |MK|\cdot|KP| = |KS|^2$. And

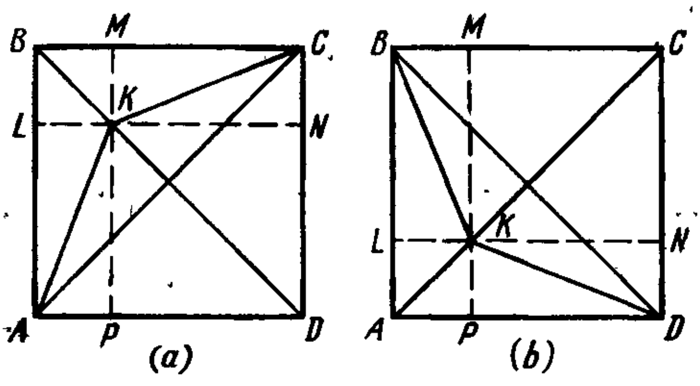


Fig. 27

since $|LK| + |KN| = |MK| + |KP| = a$, two cases are possible: either $|LK| = |KM|$, $|KP| = |KN|$, or $|LK| = |KP|$, $|MK| = |KN|$, that is, the point K lies either on the diagonal AC or BD. Consider both cases.

(1) K lies on the diagonal BD (Fig. 27, a). The figure represents the projection of the pyramid on the plane

$ABCD$. The point S is found "above" K. Setting $|LK| = |KM| = x$, we now find:

$$|KS| = \sqrt{|LK| \cdot |KN|} = \sqrt{x(a-x)},$$

$$|SL| = \sqrt{|LK|^2 + |KS|^2} = \sqrt{ax},$$

$$S_{ABS} = \frac{a\sqrt{ax}}{2}.$$

Analogously, $S_{ADS} = \dfrac{a\sqrt{a(a-x)}}{2}$. Further, $V_{ABDS} = \dfrac{1}{6}a^2\sqrt{x(a-x)}$. On the other hand, by the formula of Problem 11, we have

$$V_{ABDS} = \frac{2}{3}\frac{S_{ABS}S_{BDS}\sin\alpha}{|AK|}$$

$$= \frac{a^3\sqrt{x(a-x)}\sin\alpha}{6\sqrt{(a-x)^2 + x^2 + x(a-x)}}.$$

Equating two expressions for V_{ABDS}, we get

$$x^2 - ax + a^2\cos^2\alpha = 0,$$

whence $x(a-x) = a^2\cos^2\alpha$,

$$V_{ABCDS} = \frac{a^3|\cos\alpha|}{3}.$$

The problem has a solution if $|\cos\alpha| \leqslant \frac{1}{2}$. Besides, the angle at the edge AS is obtuse, since the plane ASM is perpendicular to the face ASD, and this plane passes inside the dihedral angle between the planes ASB and ASD. Consequently, in the first case the problem has a solution if $\frac{\pi}{2} < \alpha \leqslant \frac{2\pi}{3}$.

(2) The point K lies on the diagonal AC (Fig. 27, b). Reasoning as in Case (1), we get (as before, $|LK| = x$):

$$V_{ABDS} = \frac{a^2\sqrt{x(a-x)}}{6} = \frac{a^3 x \sin\alpha}{6\sqrt{x(x+x)}},$$

whence we easily find $x = a\,|\cos\alpha|$,

$$V = \frac{a^3 \sqrt{|\cos\alpha|\,(1 - |\cos\alpha|)}}{6}.$$

The same as in the first case, $\alpha > \frac{\pi}{2}$. Thus, we get the answer.

Answer: if $\frac{\pi}{2} < \alpha \leqslant \frac{2\pi}{3}$, two answers are possible:

$$V_1 = -\frac{a^3 \cos\alpha}{6}, \quad V_2 = \frac{a^3 \sqrt{-\cos\alpha\,(1+\cos\alpha)}}{6};$$

if $\alpha > \frac{2\pi}{3}$, $V = \frac{a^3 \sqrt{-\cos\alpha\,(1+\cos\alpha)}}{6}$.

148. Let us first solve the following problem. In the triangle ABC points L and K are taken on the sides AB and AC so that $\frac{|AL|}{|LB|} = m$, $\frac{|AK|}{|KC|} = n$. What is the ratio in which the median AM is divided by the line KL?

Denote by N the point of intersection of KL and AM; Q is the point of intersection of KL and BC, P is the point of intersection of KL and the straight line parallel to BC and passing through A. Let $|BC| = 2a$, $|QC| = b$, $|AP| = c$, $n > m$. Then, from the similarity of the corresponding triangles we shall have: $\frac{c}{b} = n$, $\frac{c}{b+2a} = m$, whence $\frac{|AN|}{|NM|} = \frac{c}{b+a} = \frac{2mn}{m+n}$.

Let now m, n, and p be the ratios in which the edges AB, AC, and AD are divided by the plane. To determine them, we shall have the following system:

$$\frac{2mn}{m+n} = 2, \quad \frac{2np}{n+p} = \frac{1}{2}, \quad \frac{2pm}{p+m} = 4,$$

whence

$$m = -\frac{4}{5}, \quad n = \frac{4}{9}, \quad p = \frac{4}{7}.$$

The fact that $-1 < m < 0$ means that the point L lies on the extension of AB beyond the point A, that is, our plane intersects the edges AC, AD, BC, and BD. Further, determining the ratios in which the edges BC and BD are divided $\left(\text{we shall get } \frac{5}{7} \text{ and } \frac{5}{9}\right)$, we find the answer: $\frac{7123}{16{,}901}$.

149. Consider the pyramid $SABC$ (Fig. 28) in which $|CA| = |AB|$, $\widehat{BAC} = \frac{2\pi}{n}$, SA is perpendicular to the plane ABC, and such that the vertex A is projected on

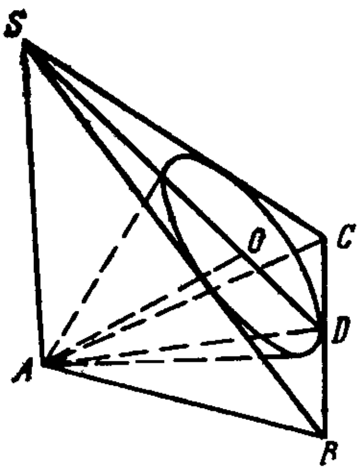


Fig. 28

the plane SBC into the point O which is the centre of the circle inscribed in SBC.

Let us inscribe a cone in this pyramid so that its vertex coincides with A, and the circle of its base is represented by the circle inscribed in SBC. It is obvious that if we take n such pyramids whose bases lie in the plane ABC so that their bases congruent to the triangle ABC form a regular n-gon with centre at A, then the cones inscribed in these pyramids form the desired system of cones.

Further, let D be the midpoint of BC, $|OD| = r$, $|AD| = l$. Then $|SD| = \frac{l^2}{r}$, $|BD| = l \tan \frac{\pi}{n}$. Since

$$\widehat{SBD} = 2\widehat{OBD}, \qquad \tan \widehat{SBD} = \frac{|SD|}{|BD|} = \frac{l}{r \tan \frac{\pi}{n}},$$

$\tan \widehat{OBD} = \dfrac{r}{l \tan \frac{\pi}{n}}$, we may write the equation

$$\frac{l}{r \tan \frac{\pi}{n}} = \frac{2 \dfrac{r}{l \tan \frac{\pi}{n}}}{1 - \dfrac{r^2}{l^2 \tan^2 \frac{\pi}{n}}},$$

whence $\dfrac{r}{l} = \dfrac{\tan \frac{\pi}{n}}{\sqrt{1 + 2 \tan^2 \frac{\pi}{n}}}$.

Answer: $2 \arcsin \dfrac{\tan \frac{\pi}{n}}{1 + 2 \tan^2 \frac{\pi}{n}}$.

150. Let the plane AKN touch the ball at the point P, and the straight line AP intersect NK at the point M

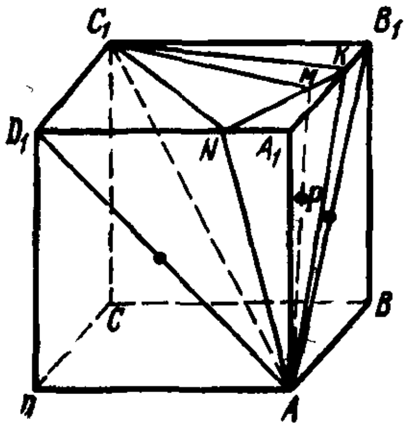


Fig. 29

(Fig. 29). Then the plane C_1NA is the bisector plane of the dihedral angle formed by the planes D_1C_1A and C_1MA (the planes D_1AN and ANM touch the ball, and the planes D_1C_1A and C_1MA pass through its centre). In the same way, the plane C_1KA is the bisector plane of the dihedral angle formed by the planes MC_1A and C_1B_1A. Thus, the dihedral angle between the planes AC_1K and AC_1N is one-half the dihedral angle between the planes AD_1C_1 and AB_1C_1 equal to $2\pi/3$.

Answer: $\pi/3$.

151. Let K, L, and M be the midpoints of the edges AB, AC, and AD (Fig. 30). From the conditions of the

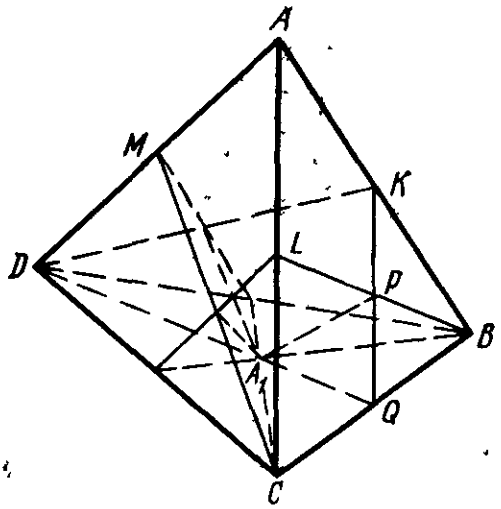


Fig. 30

problem it then follows that the tetrahedron $A_1B_1C_1D_1$ is bounded by the planes DKA_1, BLA_1, CMA_1, and the plane passing through A parallel to BCD. And the vertices B_1, C_1, and D_1 are arranged so that the points M, K, and L are the midpoints of CB_1, DC_1, and BD_1 (the points B_1, C_1, and D_1 are not shown in the figure).

Let now Q be the midpoint of BC, P the point of intersection of BL and KQ. To find the volume of the common part of two pyramids $ABCD$ and $A_1B_1C_1D_1$, we must from the volume V of the pyramid $ABCD$ subtract the volumes of three pyramids equivalent to $DKBQ$ (each of them has the volume equal to $\frac{1}{4}V$), and add the volumes of three pyramids equivalent to A_1BQP. The volume of the last pyramid is equal to $\frac{1}{24}V$. Thus, the volume of the common part is equal to $\frac{3}{8}V$.

152. Let us first prove that the dihedral angles at the edges DB and AC are equal to $\pi/2$ (each). Let $|AD| = |CD| = |BC| = a$, $|BD| = |AC| = b$, $|AB| = c$,

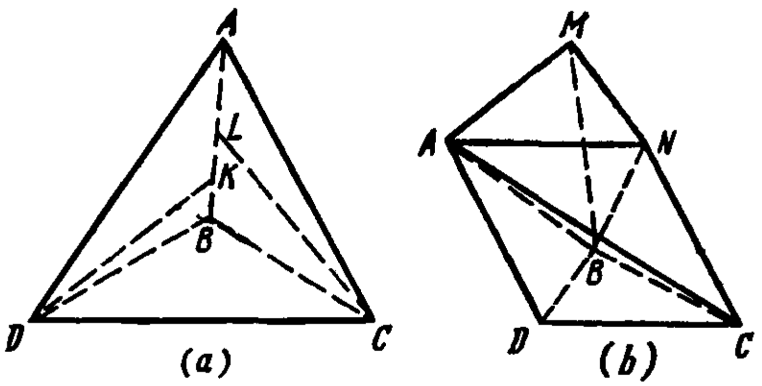


Fig. 31

$b > a$. From D and C drop perpendiculars DK and CL on the edge AB (Fig. 31, a). Let us introduce the following notation:

$$|AK| = |BL| = x, \ |KL| = |c - 2x|, \ |DK| = |CL| = h.$$

Since the dihedral angle at the edge AB is equal to $\pi/3$, we have $|DC|^2 = |DK|^2 + |CL|^2 - |DK| \times |CL| + |KL|^2$, that is, $a^2 = h^2 + (c-2x)^2$. Replacing h^2 by $a^2 - x^2$, we get $3x^2 - 4cx + c^2 = 0$, whence $x_1 = c/3$, $x_2 = c$. From the condition $b > a$ it follows that $x < c/2$, hence $x = c/3$. Thus, the quantities a, b, and c are related as follows: $c^2 = 3(b^2 - a^2)$.

Find the areas of the triangles ABD and ACD:

$$S_{ABD} = S_{ABC} = \frac{1}{2}c\sqrt{a^2 - \frac{c^2}{9}} = \frac{1}{2}c\sqrt{\frac{4a^2 - b^2}{3}},$$

$$S_{ACD} = S_{BDC} = \frac{1}{4}b\sqrt{4a^2 - b^2}.$$

Expressing the volume of the tetrahedron $ABCD$ by the formula of Problem 11 in terms of the dihedral angle at the edge AB and the areas of the faces ABD and ABC, and then in terms of φ the dihedral angle at the edge AC (it is also equal to the angle at the edge BD) and the areas of the faces ABC and ACD, we get

$$V_{ABCD}=\frac{1}{3}\,\frac{S_{ABD}S_{ABC}}{|AB|}\cdot\frac{\sqrt{3}}{2}=\frac{1}{3}\,\frac{S_{ACD}S_{ABC}}{|AC|}\sin\varphi,$$

whence

$$\sin\varphi=\frac{S_{ABD}}{S_{ACD}}\,\frac{|AC|}{|AB|}\cdot\frac{\sqrt{3}}{2}$$

$$=\frac{2c\sqrt{\dfrac{4a^2-b^2}{3}}}{b\sqrt{4a^2-b^2}}\cdot\frac{b}{c}\cdot\frac{\sqrt{3}}{2}=1.$$

Hence, $\varphi=\dfrac{\pi}{2}$.

To determine the sum of the remaining three dihedral angles, consider the prism $BCDMNA$ (Fig. 31, b). The tetrahedron $ABCN$ is congruent to the tetrahedron $ABCD$, since the plane ABC is perpendicular to the plane of $ADCN$, but $ADCN$ is a rhombus, consequently, the tetrahedra $ABCD$ and $ABCN$ are symmetric with respect to the plane BCA. Just in the same way the tetrahedron $ABMN$ is symmetric to the tetrahedron $ABCN$ with respect to the plane ABN (the angle at the edge BN in the tetrahedron $ABCN$ is congruent to the angle at the edge BD of the tetrahedron $ABCD$, that is, equal to $\pi/2$), consequently, the tetrahedron $ABMN$ is congruent to the tetrahedron $ABCN$ and is congruent to the original tetrahedron $ABCD$.

The dihedral angles of the prism at the edges CN and BM are respectively congruent to the dihedral angles at the edges DC and BC of the tetrahedron $ABCD$. And since the sum of the dihedral angles at the lateral edges of the triangular prism is equal to π, the sum of the dihedral angles at the edges AD, DC, and CB of the tetrahedron $ABCD$ is also equal to π, and the sum of all the dihedral angles of the tetrahedron excluding the given angle at the edge AB is equal to 2π.

153. Let in the triangle ABC the sides BC, CA, and AB be respectively equal to a, b, and c. Since the pyramids $ABCC_1$, ABB_1C_1, and $AA_1B_1C_1$ are congruent, it follows that each of them has two faces congruent to the triangle ABC. Indeed, if each pyramid had only one such face, then between the vertices of the pyramids $ABCC_1$ and $A_1B_1C_1A$ there would be the correspondence $A \to A_1$,

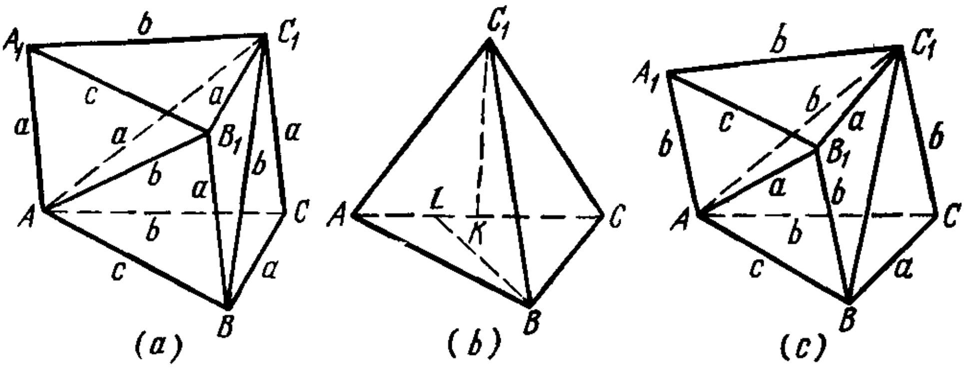


Fig. 32

$B \to B_1$, $C \to C_1$, $C_1 \to A$, that is, $|CC_1| = |AC_1|$, $|BC_1| = |B_1A|$, and this would mean that none of the faces in the pyramid ABC_1B_1 is equal to the triangle ABC. Now, it is easy to conclude that the lateral edge of the prism is equal to a, or b, or c (if, for instance, the triangle AC_1B is congruent to the triangle ABC, then the face A_1B_1A in the pyramid $A_1B_1C_1A$ corresponds to the face AC_1B of the pyramid $ABCC_1$ and the triangle A_1B_1A is congruent to the triangle ABC).

Consider all possible cases.

(1) $|AA_1| = |BB_1| = |CC_1| = a$ (Fig. 32, *a*). Then from the vertex C of the pyramid $ABCC_1$ two edges of length a and one edge of length b emanate, and an edge of length c lies opposite the edge CC_1. Hence it follows that to the vertex C of the pyramid $ABCC_1$ there must correspond the vertex C_1 of the pyramid $A_1B_1C_1A$ and $|AC_1| = a$. Now it is possible to conclude that $|AB_1| = |BC_1| = b$.

In all the three pyramids, the dihedral angles at the edges of length b are congruent, the sum of two such angles being equal to π (for instance, two angles at the edge C_1B in the pyramids $ABCC_1$ and ABB_1C_1), that is, each of them is equal to $\pi/2$.

Draw perpendiculars BL and C_1K to the edge AC (Fig. 32, b). Since the dihedral angle at the edge AC is equal to 90°, we have

$$b^2 = |C_1B|^2 = |C_1K|^2 + |KL|^2 + |LB|^2$$
$$= |C_1C|^2 - |KC|^2 + (|KC| - |LC|)^2 + |BC|^2$$
$$-|LC|^2 = 2a^2 - bx,$$

where $x = |LC|$, and is found from the equation

$$a^2 - x^2 = c^2 - (b-x)^2, \quad x = \frac{a^2+b^2-c^2}{2b}.$$

Thus, $3a^2 - 3b^2 + c^2 = 0$. But, by the hypothesis, ABC is a right triangle. This is possible only under the condition $c^2 = a^2 + b^2$. Consequently, $b = a\sqrt{2}$, $c = a\sqrt{3}$.

Now, it is possible to find the dihedral angle at the edge BC of our prism. $\widehat{ACC_1} = \pi/4$ is the linear angle of this dihedral angle (ABC and C_1CB are right triangles with right angles at the vertex C). The dihedral angle at the edge AB of the pyramid $ABCC_1$ is equal to $\pi/3$. Let us show this. Let this angle be equal to φ. Then the dihedral angle at the edge AB of the prism $ABCA_1B_1C_1$ is equal to 2φ, and at the edge A_1B_1 to φ. Thus,

$$3\varphi = \pi, \quad \varphi = \frac{\pi}{3}.$$

(2) $|AA_1| = |BB_1| = |CC_1| = b$ (Fig. 32, c). In this case, in the pyramid $ABCC_1$ two edges of length b and one edge of length a emanate from the vertex C. Hence, the pyramid $A_1B_1C_1A$ has also such a vertex. It can be either the vertex A or C_1. In both cases we get $|AB_1| = a$, $|AC_1| = b$ (we remind here that two faces with sides a, b, and c must be found). Thus, each of the pyramids $ABCC_1$ and $A_1B_1C_1A$ has one face representing a regular triangle with side b, while the pyramid ABB_1C_1 has not such a face whatever the length of the edge BC_1 is. Thus, this case is impossible.

(3) $|AA_1| = |BB_1| = |CC_1| = c$. This case actually coincides with the first, only the bases ABC and $A_1B_1C_1$ are interchanged.

Answer: $\frac{\pi}{2}$, $\frac{\pi}{4}$ $\left(\text{or } \frac{3\pi}{4}\right)$, $\frac{\pi}{3}$ $\left(\text{or } \frac{2\pi}{3}\right)$.

154. Drop perpendiculars A_1M and B_1M on CD, B_1N and C_1N on AD, C_1K and D_1K on AB, D_1L and A_1L on CB.

Since

$$\frac{|A_1M|}{|B_1M|} = \frac{|B_1N|}{|NC_1|} = \frac{|C_1K|}{|KD_1|} = \frac{|D_1L|}{|A_1L|} = \frac{1}{3}$$

(these ratios are equal to the cosine of the dihedral angle at the edges of the tetrahedron) and $|A_1B_1| = |B_1C_1| =$

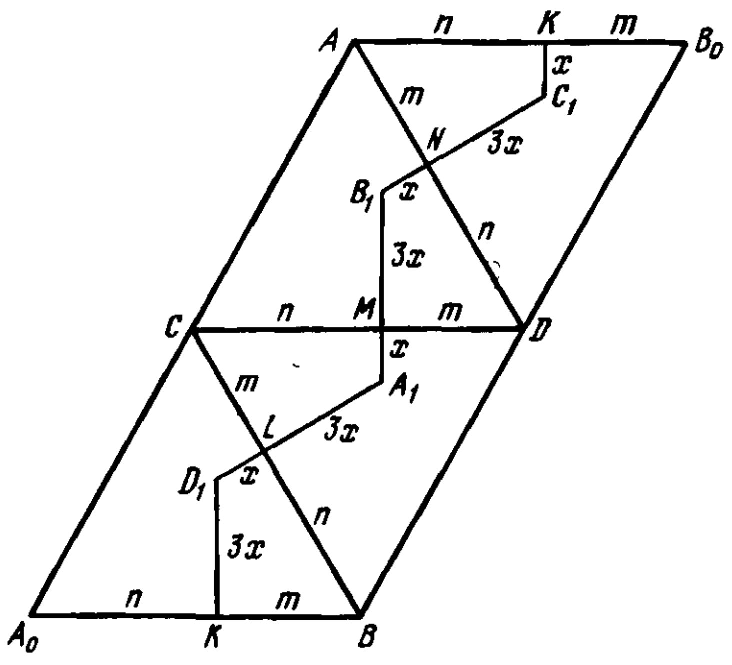


Fig. 33

$|C_1D_1| = |D_1A_1|$, the following equalities must be fulfilled: $|A_1M| = |B_1N| = |C_1K| = |D_1L| = x$, $|B_1M| = |NC_1| = |KD_1| = |A_1L| = 3x$ (Fig. 33 represents the development of the tetrahedron). Each of the edges CD, DA, AB, and BC will turn out to be divided into line segments m and n as is shown in the figure. Bearing in mind that $m+n=a$, we find $x=\frac{a\sqrt{3}}{12}$, $m=\frac{5a}{12}$, $n=\frac{7a}{12}$, and then find the volume of the tetrahedron $A_1B_1C_1D_1$.

Answer: $\frac{a^3\sqrt{2}}{162}$.

155. Without loss of generality, we will regard that all the elements of the cone tangent to the balls are in

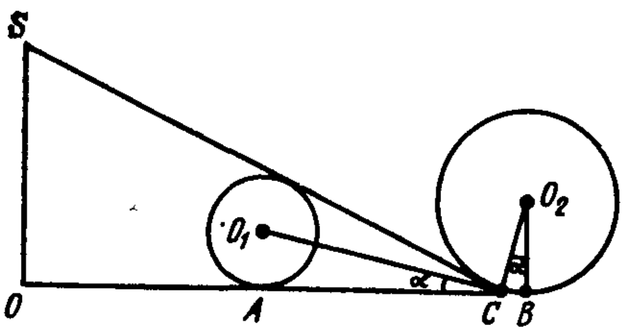


Fig. 34

contact simultaneously with two balls: inner and outer. Let us pass a section through the vertex S of the cone and the centres of the two balls touching one element (Fig. 34, the notation is clear from the figure). From the condition that n balls of radius R touch one another there follows the equality $|OA|=\frac{R}{\sin\frac{\pi}{n}}$, analogously,

$|OB|=\frac{2R}{\sin\frac{\pi}{n}}$. Consequently $|AB|=a=\frac{R}{\sin\frac{\pi}{n}}$.

Let $|AC|=x$. Then $\tan\alpha=\frac{R}{x}$, $\cot\alpha=\frac{2R}{a-x}$. Multiplying these equalities, we get the equation for x:

$x^2-ax+2R^2=0$,

whence $x_1 = \frac{a - \sqrt{a^2 - 8R^2}}{2}$, $x_2 = \frac{a + \sqrt{a^2 - 8R^2}}{2}$,

where $a = \frac{R}{\sin \frac{\pi}{n}}$.

The condition $a^2 - 8R^2 \geqslant 0$ yields the inequality $\sin \frac{\pi}{n} \leqslant \frac{1}{2\sqrt{2}}$. Besides, there must be fulfilled the inequality $\tan \alpha = \frac{R}{x} < 1$. Now, it is not difficult to obtain that the root x_1 fits if $\frac{1}{3} < \sin \frac{\pi}{n} \leqslant \frac{1}{2\sqrt{2}}$. For the root x_2 it remains one restriction: $\sin \frac{\pi}{n} \leqslant \frac{1}{2\sqrt{2}}$.

It is possible to prove that $\frac{1}{3} < \sin \frac{\pi}{n} \leqslant \frac{1}{2\sqrt{2}}$ only for $n = 9$.

The volume of the cone will be equal to $\frac{1}{3} \pi (a + x)^3 \tan 2\alpha$. Expressing a, x, and $\tan 2\alpha$ in terms of R and n by the appropriate formulas, we get the answer.

Answer:

$$V = \frac{\pi R^3 \left(3 + \sqrt{1 - 8\sin^2 \frac{\pi}{n}}\right)^3 \left(1 + \sqrt{1 - 8\sin^2 \frac{\pi}{n}}\right)}{12 \sin^2 \frac{\pi}{n} \left(1 - 6\sin^2 \frac{\pi}{n} + \sqrt{1 - 8\sin^2 \frac{\pi}{n}}\right)},$$

$$n \geqslant 9.$$

Besides, for $n = 9$ one more value is possible:

$$\frac{\pi R^3 \left(3 - \sqrt{1 - 8\sin^2 \frac{\pi}{9}}\right)^3 \left(1 - \sqrt{1 - 8\sin^2 \frac{\pi}{9}}\right)}{12 \sin^2 \frac{\pi}{9} \left(1 - 6\sin^2 \frac{\pi}{9} - \sqrt{1 - 8\sin^2 \frac{\pi}{9}}\right)}.$$

156. Projecting the cube on the plane perpendicular to B_1D, we get a regular hexagon $ABCC_1D_1A_1$ (Fig. 35) with side $\sqrt{\frac{2}{3}}\,a=b$, where a is the edge of the cube (the regular triangle BC_1A_1 will be projected into a congruent triangle, since the plane of BC_1A_1 is perpendicular to

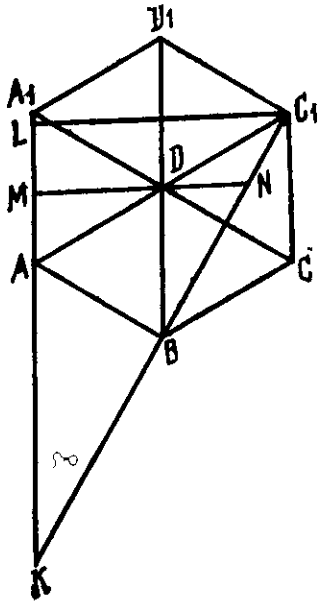


Fig. 135

B_1D). Consider the triangle KAC_1, where $|KA| = |AC_1| = 2b$, the line NM passes through the midpoint of AC_1. Let $\frac{|AM|}{|AA_1|} = x$. We then draw C_1L parallel to MN. We have:

$$|ML| = |AM|,$$

$$\frac{|KN|}{|KC_1|} = \frac{|KM|}{|KL|} = \frac{2+x}{2+2x},$$

whence

$$\frac{|BN|}{|BC_1|} = \frac{2(|KN|-|BC|)}{|KC_1|}$$

$$=2\frac{|KN|}{|KC_1|}-1=\frac{2+x}{1+x}-1=\frac{1}{1+x}.$$

Thus,

$$\frac{|BC_1|}{|BN|}-\frac{|AM|}{|AA_1|}=1+x-x=1.$$

157. If two noncongruent and similar triangles have two equal sides, then it is easy to make sure that the sides of each of them form a geometric progression, and the sides of one of them may be designated by a, λa, $\lambda^2 a$ and those of the other by λa, $\lambda^2 a$, $\lambda^3 a$.

Further, if the sides of a triangle form a geometric progression and two of them are equal to 3 and 5, then the third side will be equal to $\sqrt{15}$ (in other cases the sum of two sides will be less than the third one). Now, it is easy to prove that in our tetrahedron two faces are triangles with sides 3, $\sqrt{15}$, 5 and two other faces have sides $\sqrt{15}$, 5, $5\sqrt{\frac{5}{3}}$ or $3\sqrt{\frac{3}{5}}$, 3, $\sqrt{15}$; accordingly the problem has two answers: $\frac{55\sqrt{6}}{18}$ and $\frac{11}{10}\sqrt{10}$.

158. Introduce a rectangular coordinate system so that the first line coincides with the x-axis, the second line is parallel to the y-axis and passes through the point $(0, 0, a)$, and the third line is parallel to the z-axis and passes through the point $(a, a, 0)$. Let $ABCDA_1B_1C_1D_1$ be a parallelepiped in which the points A and C lie on the first line and have the coordinates $(x_1, 0, 0)$, $(x_2, 0, 0)$, respectively, the points B and C_1 on the second line, their coordinates are $(0, y_1, a)$ and $(0, y_2, a)$, and the points D and B_1 on the third line, their respective coordinates are (a, a, z_1) and (a, a, z_2). From the condition of the equality of the vectors $\overrightarrow{AD}=\overrightarrow{BC}=\overrightarrow{B_1C_1}$, we get $a-x_1=x_2=-a$, $a=-y_1=y_2-a$, $z_1=-a=a-z_2$, whence $x_1=2a$, $x_2=-a$, $y_1=-a$, $y_2=2a$, $z_1=-a$, $z_2=2a$. Thus, we have $A\ (2a, 0, 0)$, $B\ (0, -a, a)$, $C\ (-a, 0, 0)$, $D\ (a, a, -a)$, $B_1\ (a, a, 2a)$, $C_1\ (0, 2a, a)$. It is possible to check that $\overrightarrow{AB}=\overrightarrow{DC}$. Further, $|AC|=3a$, $|AB|=a\sqrt{6}$, $|BC|=a\sqrt{3}$, that is, ABC is a right triangle, hence, the area of $ABCD$ will be $|AB|\cdot|BC|=3a^2\sqrt{2}$.

The equation of the plane $ABCD$ is $y + z = 0$, hence, the distance from B_1 to this plane will be equal to $\frac{3a}{\sqrt{2}}$.

Answer: $9a^3$.

159. Consider the regular pyramid $ABCDS$ in which the section $KLMNP$ is drawn representing a regular pen-

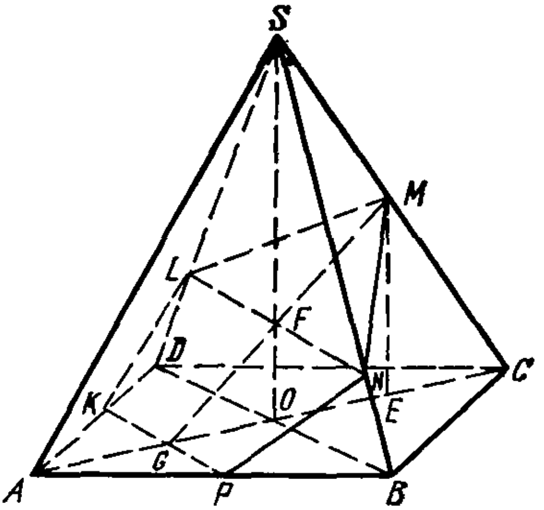


Fig. 36

tagon with side a (Fig. 36). Let the diagonal of the base of the pyramid be equal to b, and its lateral edge to l. Let us also set $|SM| = xl$, $|SN| = yl$. Since the pentagon $KLMNP$ is regular, we have

$$|LM| = 2a\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2}a = \mu a,$$

$$\frac{|MF|}{|FG|} = \frac{1-\cos\frac{2\pi}{5}}{\cos\frac{\pi}{5}+\cos\frac{2\pi}{5}} = \frac{\sqrt{5}-1}{2} = \lambda.$$

We have: $|KP| = a$, $|GO| = \frac{b-a}{2}$. On the other hand,

$$|OE| = |OC|\frac{SM}{SC} = \frac{b}{2}x, \quad |ME| = |SO|\frac{|MC|}{|SC|} =$$

$h(1-x)$, $|FO| = h(1-y)$, where h is the altitude of the pyramid, consequently,

$$\frac{|GO|}{|FO|} = \frac{|OE|}{|ME| - |FO|}, \quad |GO| = \frac{(1-y)\,xb}{2(y-x)}.$$

Equating the found expressions for $|GO|$, we get the equation

$$\frac{(1-y)\,xb}{y-x} = b-a. \tag{1}$$

Further

$$\frac{|OE|}{|GO|} = \frac{|MF|}{|FG|} = \lambda,$$

whence

$$\frac{y-x}{1-y} = \lambda. \tag{2}$$

Since $|LN| = \mu a$, $|LN| = y\,|DB|$, we have

$$yb = \mu a. \tag{3}$$

And, finally, consider the triangle PNB in which $|PN| = a$, $|NB| = (1-y)\,l$, $|PB| = \frac{b-a}{2}\sqrt{2}$, $\cos\widehat{PBN} = \cos\widehat{ABS} = \frac{b}{2\sqrt{2}l}$.

By the theorem of cosines, we get

$$a^2 = (1-y)^2 l^2 + \frac{(b-a)^2}{2} - \frac{(1-y)(b-a)\,b}{2}. \tag{4}$$

Taking into consideration that $\mu = \frac{\sqrt{5}+1}{2}$, $\lambda = \frac{\sqrt{5}-1}{2}$, from Equations (1)-(3) we find $y = \frac{\sqrt{5}-1}{2}$, $b = \frac{\sqrt{5}+3}{2}a$, then from Equation (4) we get

$$l^2 = \frac{a^2(7+3\sqrt{5})}{4} = \frac{b^2}{2}.$$

Thus, the volume of the pyramid is equal to

$$\frac{1}{3}\cdot\frac{b^2}{2}\sqrt{l^2-\frac{b^2}{4}}=\frac{b^3}{12}=\frac{(9+4\sqrt{5})}{12}a^3.$$

160. We introduce the usual notation: a, b, c denote the sides of the given triangle, h_a, h_b, h_c its altitudes, p one half of its perimeter, r the radius of the inscribed circle. Let M denote the point of intersection of the planes A_1B_1C, A_1BC_1, and AB_1C_1, O_a, O_b, O_c the centres of the externally inscribed circles (O_a is the centre of the circle touching the side BC and the extensions of AB and AC, and so on). Prove that $O_aO_bO_cM$ is the desired pyramid, the altitude dropped from the point M passing through the centre of the inscribed circle (O), and $|MO| = 2r$.

Consider, for instance, the plane A_1B_1C. Let K be the point of intersection of this plane with the line AB,

$$\frac{|KA|}{|KB|}=\frac{|AA_1|}{|BB_1|}=\frac{h_a}{h_b}=\frac{b}{a}=\frac{|AC|}{|BC|},$$

that is, K is the point of intersection of the line AB and the bisector of the exterior angle C. Hence it follows that the base of our pyramid is indeed the triangle $O_aO_bO_c$ and that the point M is projected into the point O. Find $|MO|$:

$$\frac{|MO|}{h_a}=\frac{|OO_a|}{|AO_a|}=\frac{r_a-r}{r_a},$$

where r_a is the radius of the externally inscribed circle centred at O_a: $r_a=\dfrac{S}{p-a}$, $r=\dfrac{S}{p}$, $h_a=\dfrac{2S}{a}$, consequently,

$$|MO|=h_a\frac{r_a-r}{r_a}=\frac{2S}{a}\,\frac{\dfrac{1}{p-a}-\dfrac{1}{p}}{\dfrac{1}{p-a}}=\frac{2S}{p}=2r.$$

Find the area of the triangle $O_aO_bO_c$. Note that O_aA, O_bB, O_cC are the altitudes of this triangle. The angles of the triangle $O_aO_bO_c$ are found readily, for instance,

$$\widehat{O_cO_aO_b} = \widehat{BO_aC} = 180° - \left(90° - \frac{\hat{B}}{2}\right) - \left(90° - \frac{\hat{C}}{2}\right)$$
$$= 90° - \frac{\hat{A}}{2}.$$

Other angles are found in a similar way. The circle with diameter O_bO_c passes through B and C, consequently,

$$|O_bO_c| = \frac{|BC|}{\sin \widehat{BO_bC}} = \frac{a}{\sin \frac{\hat{A}}{2}},$$

exactly in the same way $|O_bO_a| = \dfrac{c}{\sin \frac{\hat{C}}{2}}$, hence

$$|O_aA| = |O_aO_b| \sin \widehat{O_aO_bA} = \frac{c}{\sin \frac{\hat{C}}{2}} \cos \frac{\hat{B}}{2}.$$

Thus, the area of the triangle $O_aO_bO_c$ (let us denote it by Q) will be

$$Q = \frac{1}{2} \frac{ac}{\sin \frac{\hat{A}}{2} \sin \frac{\hat{C}}{2}} \cos \frac{\hat{B}}{2}$$
$$= \frac{ac \sin \hat{B}}{4} \cdot \frac{1}{\sin \frac{\hat{A}}{2} \sin \frac{\hat{B}}{2} \sin \frac{\hat{C}}{2}}. \quad (1)$$

Find $\sin\frac{\hat{A}}{2}$:

$$\sin\frac{\hat{A}}{2}=\sqrt{\frac{1-\cos\hat{A}}{2}}=\sqrt{\frac{1}{2}\left(1-\frac{b^2+c^2-a^2}{2bc}\right)}$$
$$=\sqrt{\frac{(p-b)(p-c)}{bc}}.$$

Then $\sin\frac{\hat{B}}{2}$ and $\sin\frac{\hat{C}}{2}$ are found in the same way. Substituting them in (1), we get

$$Q=S\frac{abc}{2(p-a)(p-b)(p-c)},$$

and the volume of the pyramid $MO_aO_bO_c$ will be

$$V=\frac{S_{abcr}}{3(p-a)(p-b)(p-c)}=\frac{1}{3}abc=\frac{4}{3}SR.$$

Section 2

161. No, not in any.

162. The indicated property is possessed by a pyramid in which two opposite dihedral angles are obtuse.

163. Prove that if the straight line is not perpendicular to the plane and forms equal angles with two intersecting lines in this plane, then the projection of this line on the plane also makes equal angles with the same lines, that is, it is parallel to the bisector of either of the two angles made by them.

164. A triangle, a quadrilateral, and a hexagon. A cube cannot be cut in a regular pentagon, since in a section having more than three sides there is at least one pair of parallel sides, but a regular pentagon has no parallel sides.

165. On the edges of the trihedral angle lay off equal line segments SA, SB, and SC from the vertex S. Denote by O the projection of S on the plane ABC. ASB and AOB are isosceles triangles with a common base AB, the lateral sides of the triangle AOB being shorter than those of the triangle ASB. Consequently, $\widehat{AOB} > \widehat{ASB}$. Similar inequalities hold for other angles. Thus,

$$\widehat{ASB} + \widehat{BSC} + \widehat{CSA} < \widehat{AOB} + \widehat{BOC} + \widehat{COA} \leqslant 2\pi.$$

(The last sum is equal to 2π if O is inside the triangle ABC and is less than 2π if O lies outside of this triangle.)

To prove the second statement, take an arbitrary point inside the given angle and from this point drop perpendiculars on the faces of the given angle. These perpendiculars will represent the edges of another trihedral angle. (The obtained angle is called *complementary* to the given trihedral angle. This technique is a standard method in the geometry of trihedral angles.) The dihedral angles of the given trihedral angle are complemented to π by the plane angles of the complementary trihedral angle, and vice versa. If α, β, γ are the dihedral angles of the given trihedral angle, then, using the above-proved inequality for plane angles, we shall have $(\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) < 2\pi$, whence it follows that $\alpha + \beta + \gamma > \pi$.

166. (1) Let S be the vertex of the angle, M a point on an edge, M_1 and M_2 the projections of M on two other edges, N the projection of M on the opposite face. Suppose that the edge SM corresponds to the dihedral angle C. If $|SM| = a$, then, finding successively $|SM_1|$ and then from the triangle MM_1N, $|MN|$, or in a different way, first $|SM_2|$, and then from the triangle MM_2N, $|MN|$, we arrive at the equality

$$|MN| = a \sin\alpha \sin B = a \sin\beta \sin A,$$

that is,

$$\frac{\sin\alpha}{\sin A} = \frac{\sin\beta}{\sin B}.$$

(2) Denote by **a**, **b**, and **c** the unit vectors directed along the edges of the trihedral angle (**a** lies opposite the plane angle of size α, **b** opposite β, **c** opposite γ). The vector **b** can be represented in the form: $\mathbf{b} = \mathbf{a}\cos\gamma + \boldsymbol{\eta}$,

where $|\boldsymbol{\eta}| = \sin\gamma$, $\boldsymbol{\eta}$ is a vector perpendicular to $\mathbf{a}$; analogously, $\mathbf{c} = \mathbf{a}\cos\beta + \boldsymbol{\xi}$ where $|\boldsymbol{\xi}| = \sin\beta$, $\boldsymbol{\xi}$ is perpendicular to $\mathbf{a}$. The angle between the vectors $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ is equal to A.

Multiplying $\mathbf{b}$ and $\mathbf{c}$ as scalars, we get

$$\mathbf{bc} = \cos\alpha = (\mathbf{a}\cos\gamma + \boldsymbol{\eta})(\mathbf{a}\cos\beta + \boldsymbol{\xi})$$
$$= \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos A,$$

which was just required to be proved.

(3) From a point inside the angle drop perpendiculars on the faces of the given angle. We get, as is known (see Problem 165), a trihedral angle complementary to the given. The plane angles of the given trihedral angle make the dihedral angle of the complementary angle be equal to π. Applying the first theorem of cosines to the complementary trihedral angle, we get our statement.

167. Take advantage of the first theorem of cosines (see Problem 166).

168. Take advantage of the second theorem of cosines (see Problem 166).

169. The sum of all the plane angles of the tetrahedron is equal to 4π. Hence, there is a vertex the sum of plane angles at which does not exceed π. All the plane angles at this angle are acute. Otherwise, one angle would be greater than the sum of two others.

170. This property is possessed by the edge having the greatest length.

171. Let ABC be a perpendicular section, $|BC| = a$, $|CA| = b$, $|AB| = c$. Through A pass the section AB_1C_1 (B and B_1, C and C_1 lie on the corresponding edges). Let then $|BB_1| = |x|$, $|CC_1| = |y|$. (If B_1 and C_1 lie on one side from the plane ABC, then x and y have the same sign, and if on different sides, then x and y have opposite signs.) For the triangle AB_1C_1 to be regular, it is necessary and sufficient that the following equalities be fulfilled:

$$c^2 + x^2 = b^2 + y^2,$$
$$b^2 + y^2 = a^2 + (x - y)^2.$$

Let us show that this system has always a solution. Let $a \geqslant b$ and $c \geqslant b$. It is easy to show that the set of points in the (x, y)-plane satisfying the first equation and situated in Quadrant I is a line which approaches without bound the straight line $y = x$ with increasing x and for $x = 0$, $y = \sqrt{c^2 - b^2}$. (As is known, the equation $y^2 - x^2 = k$ describes an equilateral hyperbola.) Similarly, the line described by the second equation approaches the straight line $y = x/2$ with increasing x and for x tending to zero y increases without bound. (The set of points satisfying the second equation is also a hyperbola.) Hence it follows that these two lines intersect, that is, the system of equations always has a solution.

172. Denote the remaining two vertices of the tetrahedron by C and D. By the hypothesis, $|AC| + |AD| = |AB|$. Consider the square $KLMN$ with side equal to $|AB|$. On its sides LM and MN take points P and Q such that $|PM| = |AD|$, $|QM| = |AC|$. Then $|LP| = |AC|$, $|NQ| = |AD|$, $|PQ| = |DC|$ and, consequently, $\triangle KLP = \triangle ABC$, $\triangle KNQ = \triangle BAD$, $\triangle BDC = \triangle KPQ$. These equalities imply the statement of the problem.

173. No, not any. For instance, if one of the plane angles of the trihedral angle is sufficiently small and two other are right angles, then it is easy to verify that no section of this trihedral angle is a regular triangle.

174. Show that if at least one plane angle of the given trihedral angle is not equal to 90°, then it can be cut by a plane so that the section thus obtained is an obtuse triangle. And if all the plane angles of the trihedral angle are right angles, then any of its sections is an acute triangle. For this purpose, it suffices to express the sides of an arbitrary section by the Pythagorean theorem in terms of the line segments of the edges and to check that the sum of the squares of any two sides of the section is greater than the square of the third side.

175. Let a be the length of the greatest edge, b and c the lengths of the edges adjacent to one of the end points of the edge a, and e and f to the other.

We have: $(b + c - a) + (e + f - a) = b + c + e + f - 2a > 0$. Hence it follows that at least one of the following two inequalities is fulfilled: $b + c - a > 0$ or $e + f - a > 0$. Hence, the triple of the line segments a, b, c or a, e, f can form a triangle.

176. In any tetrahedron, there is a vertex for which the sum of certain two plane angles is less than 180°. (Actually, a stronger statement holds: there is a vertex at which the sum of all plane angles does not exceed 180°.) Let the vertex A possess this property. On the edge emanating from A take points K, L, M such that $\widehat{ALM} = \widehat{KAL} = \alpha$, $\widehat{ALK} = \widehat{LAM} = \beta$. It can be done if $\alpha + \beta < 180°$.

Thus,

$\triangle KAL = \triangle LAM$, $\triangle KLM = \triangle KAM$.

In the pyramid $AKLM$, the dihedral angle at the edge AK equals the angle at the edge LM, the dihedral angle at the edge AM equals the angle at the edge KL. It is easy to make sure that the tetrahedron $KLMA$ will be brought into coincidence with itself if the edge KA is brought into coincidence with LM, and the edge AM with KL.

177. Suppose that none of the plane angles of the given trihedral angle is equal to 90°. Let S be the vertex of the

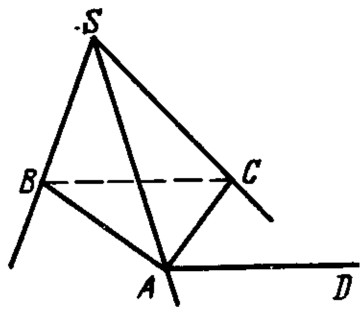


Fig. 37

given angle. Let us translate the other trihedral angle so that its vertex is brought into coincidence with a point A lying on a certain edge of the given angle (Fig. 37). AB, AC, and AD are parallel to the edges of the other dihedral angle. The points B and C are found on the edges of the given angle or on its extensions. But AB is perpendicular to SC, AC is perpendicular to SB, consequently, the projections of BS and CS on the plane ABC will be respectively perpendicular to AC and AB, that is, S is projected into the point of intersection of the altitudes of the triangle ABC, hence, AS is perpendicular to BC. Thus, the edge AD is parallel to BC, and this means that all the edges of the other trihedral angle belong to the same plane. And if one of the plane angles of the given trihedral angle is a right one, then all the edges of the other trihedral angle must lie in one face of the given angle (in one that corresponds to the right plane angle). If exactly two plane angles of the given trihedral angle are right angles, then two edges of the other trihedral angle must coincide with one edge of the given angle. Thus, the other trihedral angle can be nondegenerate only if all the plane angles of the given trihedral angle are right ones.

178. The straight line l can be regarded as the diagonal of the rectangular parallelepiped; it makes angles α, β,

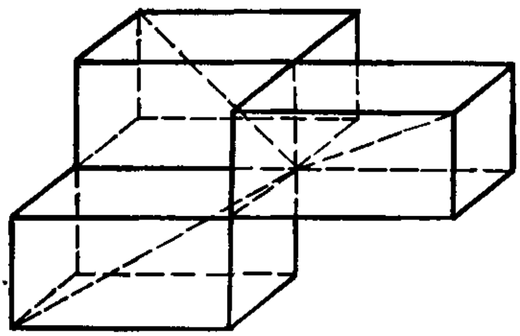


Fig. 38

and γ with edges. Then, arranging three congruent parallelepipeds in the way shown in Fig. 38, we obtain that the angles between the three diagonals of these parallelepipeds emanating from a common vertex are equal to 2α, 2β, 2γ. Consequently, $2\alpha + 2\beta + 2\gamma < 2\pi$.

179. Let S be the vertex of the angle, A, B, and C certain points on its edges. Let us prove that the angle between any edge and the plane of the opposite face is always less than either of the two plane angles including this edge. Since an angle between a straight line and a plane cannot be obtuse, it suffices to consider the case when the plane angles adjacent to the edge are acute.

Let A_1 be the projection of A on the face SBC, A_2 the projection of A on the edge SB, since $|SA_2| \geqslant |SA_1|$, $\widehat{ASA_1} \leqslant \widehat{ASA_2} = \widehat{ASB}$ (remember that all the plane angles at the vertex S are acute). From here readily follows the first part of our problem.

Let us prove the second part. We have: $\widehat{ASB}-\widehat{BSA_1}\leqslant \widehat{ASA_1}$, $\widehat{ASC}-\widehat{CSA_1}\leqslant \widehat{ASA_1}$, (at least one inequality is strict). Adding together these inequalities, we get

$$\widehat{ASB}+\widehat{ASC}-\widehat{CSB}<2\widehat{ASA_1}.$$

Writing similar inequalities for each edge and adding them, we obtain our statement. Taking a trihedral angle all the plane angles of which are obtuse and their sum is close to 2π, we make sure that in this case the statement of the second part will not be true.

180. Let α and α_1, β and β_1, γ and γ_1 be dihedral angles of the tetrahedron (the angles corresponding to opposite edges are denoted by one and the same letter). Consider four vectors a, b, c, and d perpendicular to the faces of the tetrahedron, directed outwards with respect to the tetrahedron, and having lengths numerically equal to the areas of the corresponding faces. The sum of these vectors is equal to zero. (We can give the following interpretation of this statement. Consider the vessel having the shape of our tetrahedron and filled with gas. The force of pressure on each face represents a vector perpendicular to this face and with the length proportional to its area. It is obvious that the sum of these vectors is equal to zero.) The angle between any two vectors complements to π the corresponding dihedral angle of the tetrahedron. Applying these vectors to one another in a different order, we will obtain various three-dimensional quadrilaterals. The angles of each quadrilateral are equal to the corresponding dihedral angles of the tetrahedron (two opposite angles are excluded). But the sum of angles of a space quadrilateral is less than 2π. Indeed, draw a diagonal of this quadrilateral to separate it into two triangles. The sum of angles of these triangles is equal to 2π, whereas the sum of angles of the quadrilateral is less than the sum of angles of these triangles, since in any trihedral angle a plane angle is less than the sum of two others. Thus, we have proved that the following three inequalities are fulfilled: $\alpha+\alpha_1+\beta+\beta_1<2\pi$, $\beta+\beta_1+\gamma+\gamma_1<2\pi$, $\gamma+\gamma_1+\alpha+\alpha_1<2\pi$. (Thus, we have proved the first part of the problem.) Adding these inequalities, we get $\alpha+\alpha_1+\beta+\beta_1+$

$\gamma + \gamma_1 < 3\pi$. To complete our proof, let us note that the sum of dihedral angles in any trihedral angle is greater than π (see Problem 165).

Adding up the inequalities corresponding to each vertex of the tetrahedron, we complete the proof.

Remark. In solving this problem, we have used the method consisting in that instead of the given trihedral angle, we have considered another trihedral angle whose edges are perpendicular to the edges of the given angle. The pair of trihedral angles thus obtained possesses the following property: the plane angles of one of them complement the dihedral angles of the other to π. Such angles are said to be *complementary* or *polar*. This method is widely used in spherical geometry. It was also used for solving Problem 165.

181. The statement of the problem follows from the fact that for a regular polygon the sum of the distances from an arbitrary point inside it to its sides is a constant.

182. If S_1, S_2, S_3, and S_4 denote the areas of the corresponding faces of the tetrahedron, V its volume, then

$$\frac{x_1}{h_1}+\frac{x_2}{h_2}+\frac{x_3}{h_3}+\frac{x_4}{h_4}=\frac{S_1x_1}{S_1h_1}+\frac{S_2x_2}{S_2h_2}+\frac{S_3x_3}{S_3h_3}+\frac{S_4x_4}{S_4h_4}$$
$$=\frac{S_1x_1+S_2x_2+S_3x_3+S_4x_4}{3V}=1.$$

183. Let M and K denote the midpoint of the edges AB and DC of the tetrahedron $ABCD$. The plane passing through M and K cuts the edges AD and BC at points L and N (Fig. 39, a). Since the plane DMC divides the volume of the tetrahedron into two equal parts, it suffices to prove that the pyramids $DLKM$ and $KCMN$ are equivalent. The ratio of the volume of the pyramid $KCMN$ to the volume of the entire tetrahedron $ABCD$ is equal to $\frac{1}{4}\frac{|CN|}{|CB|}$. Analogously, for the pyramid $DLKM$ this ratio is equal to $\frac{1}{4}\frac{|DL|}{|DA|}$. Hence, we have to prove the equality:

$$\frac{|DL|}{|DA|}=\frac{|CN|}{|CB|}.$$

Let us project our tetrahedron on the plane perpendicular to the line KM. The tetrahedron $ABCD$ will be projected in a parallelogram with diagonals AB and CD (Fig. 39, *b*). The line LN will pass through the point of intersection of its diagonals, consequently, our statement is true.

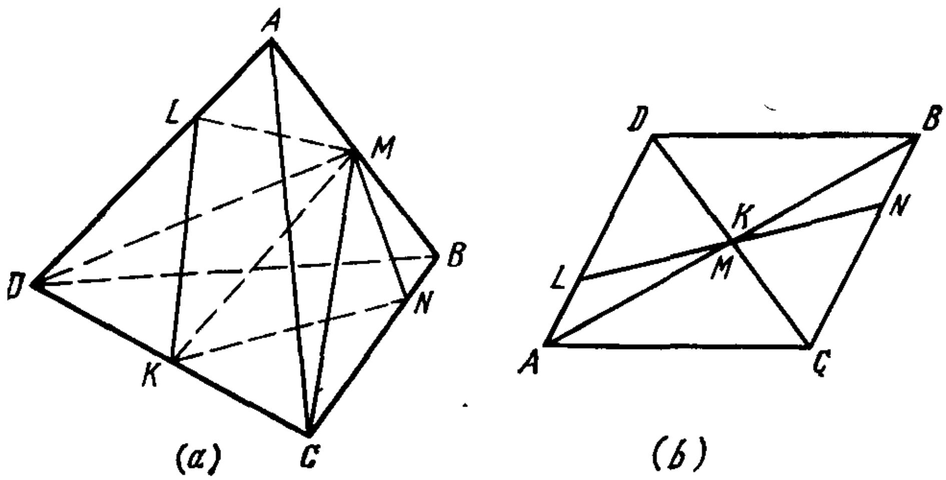


Fig. 39

184. Let for the sake of definiteness $|DA| \leqslant |DB| \leqslant |DC|$, and at least one of the inequalities is strict. Let us superpose the triangles DAB, DBC, and DCA so as to bring to a coincidence equal angles and equal sides (Fig. 40).

In the figure, the vertices of the second triangle have the subscript 1, those of the third triangle the subscript 2. But $|D_2A_2| = |DA| < |D_1C_1|$ (by the hypothesis). Consequently, $\widehat{D_2D_1B}$ is acute and $\widehat{BD_1D}$ is obtuse and $|DB| > |D_1C_1|$ which is just a contradiotion.

185. Through each edge of the tetrahedron pass a plane parallel to the opposite edge. Three pairs of planes thus obtained form a parallelepiped. Opposite edges of the tetrahedron will serve as diagonals of a pair of opposite faces of the parallelepiped. Let, for instance, a and a_1 denote the diagonals of two opposite faces of the parallelepiped, m and n their sides ($m \geqslant n$). Then $a_1a_2 \cos \alpha =$

$m^2 - n^2$. Writing such equalities for each pair of opposite edges, we will prove our statement.

186. Let the sphere pass through the vertices A, B, and C and intersect the edges DA, DB, and DC at points

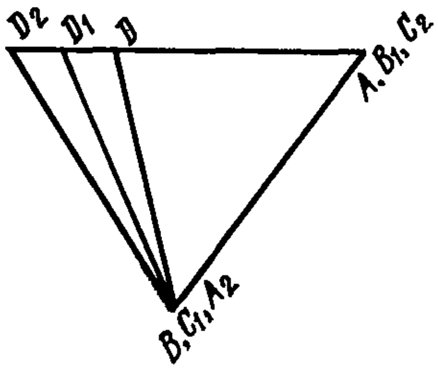


Fig. 40

K, L, and M. From the similarity of the triangles DKL and ABD, we find: $|LK| = |AB| \dfrac{|DL|}{|DA|}$ and from the similarity of the triangles DML and DBC: $|ML| = |BC| \dfrac{|DL|}{|CD|}$. But $|AB| \cdot |CD| = |BC| \cdot |BD| = 2S_{ABC}$.

Now, it is easy to make sure that $|LK| = |ML|$.

Remark. The statement of our problem will be true for any tetrahedron in which the products of opposite edges are equal.

187. The fact that the points K, L, P, and N belong to the same plane (coplanarity) implies that

$$V_{MKLP} + V_{MPNK} = V_{MNKL} + V_{MLPN}. \qquad (1)$$

From Problem 9 it follows that

$$V_{MKLP} = \frac{|MK| \cdot |ML| \cdot |MP|}{|MA| \cdot |MB| \cdot |MC|} V_{MABC},$$

$$V_{MPNK} = \frac{|MP| \cdot |MN| \cdot |MK|}{|MC| \cdot |MD| \cdot |MA|} V_{MADC},$$

$$V_{MNLK} = \frac{|MN| \cdot |ML| \cdot |MK|}{|MD| \cdot |MA| \cdot |MB|} V_{MABD},$$

$$V_{MLPN} = \frac{|ML|\cdot|MP|\cdot|MN|}{|MB|\cdot|MC|\cdot||MD|} V_{MBCD}.$$

Substituting these expressions for the corresponding quantities in (1), dividing by $|MK|\cdot|ML|\cdot|MP|\times|MN|$, multiplying by $|MA|\cdot|MB|\cdot|MC|\cdot|MD|$, expressing the volume of each of the remaining pyramids in terms of the area of the base and altitude h, we will get after the reduction by $h/3$ the statement of our problem.

188. Prove that the straight line passing through the given point parallel to a diagonal of the cube will touch each ball.

189. Both items follow from the following general statement: if the sum $\alpha|AM| + \beta|BN| + \gamma|CL|$, where α, β, γ are given coefficients, is constant, then the plane MNL passes through the fixed point. This statement, in turn, follows from the equality

$$\alpha|AM| + \beta|BN| = (\alpha + \beta)|PQ|,$$

where P is a point on AB, Q on MN,

$$\frac{|AP|}{|PB|} = \frac{|MQ|}{|QN|} = \frac{\beta}{\alpha}.$$

190. If in the tetrahedron $ABCD$ the equality $|AB| + |CD| = |BC| + |DA|$ is fulfilled, then, the same as it is done in the two-dimensional case, it is possible to prove that there is a ball touching the edges AB, BC, CD, DA, all the points of tangency being inside the line segments AB, BC, CD, and DA. If through the centre of the ball and some edge a plane is passed, then each of the dihedral angles under consideration will be divided into two parts, and for each part of any dihedral angle there is a part of the neighbouring angle which turns out to be equal to it. For instance, the angle between the planes OAB and ABC is equal to the angle between the planes OBC and ABC.

191. Let R denote the point of intersection of OM with the plane KLN (Fig. 41). The assertion that R is the centre of gravity (centroid) of the triangle KLN is equivalent to the assertion that the volumes of the tetrahedrons $MKLO$, $MLNO$, and $MNKO$ are equal. Denote by x, y, z the distances from M to the corresponding sides of the triangle ABC. Since the plane KLM is perpendicular to the edge AD, the distance from O to KLM is equal to the projection of OM on AD which is equal to the projection of MP on AD, where P is the foot of the

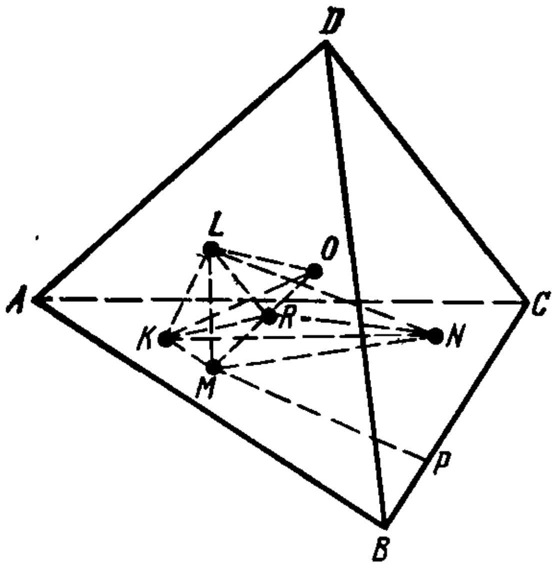


Fig. 41

perpendicular dropped from M on BC. It is easily seen that the projection of MP on AD equals $\frac{z}{\sqrt{3}}$, where z is the distance from M to BC. If α is a dihedral angle between the faces of the tetrahedron $ABCD$, then

$$V_{KLMO}=\frac{1}{6}\,|KM|\cdot|ML|\sin\alpha\cdot\frac{z}{\sqrt{3}}=\frac{xyz\sqrt{2}}{27}.$$

Each of the two other tetrahedrons $MLNO$ and $MNKO$ will have the same volume.

192. Project the tetrahedron on the plane passing through N perpendicular to CN. Let A_1, B_1, D_1, K_1, and M_1 denote the projections of the points A, B, D, K, and M. The distance between BK and CN will be equal to the distance from the point N to B_1K_1, just in the same way, the distance between AM and CN is equal to the distance from N to A_1M_1. But $A_1D_1B_1$ is an isosceles tri-

angle. The line A_1M_1 passes through K_1 (K_1 is the point of intersection of the medians). And since the triangle $A_1K_1B_1$ is also isosceles, N is usually distant from A_1K_1 and B_1K_1.

193. Let A denote a vertex of the base of the pyramid, B a point in the plane of a lateral face, $|AB| = a$, B_1 the projection of B on a side of the base, B_2 the projection of B on the plane of the base, B_3 the projection

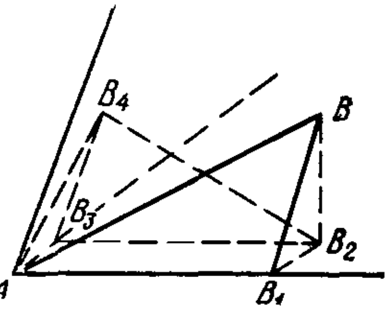


Fig. 42

of B_2 on the edge of the base adjacent to AB_1, B_4 the projection of B_2 on the lateral face adjacent to the face containing AB (Fig. 42). If now α is a dihedral angle at the base of the pyramid, $\widehat{BAB_1} = \varphi$, then

$$|B_2B_3| = |AB_1| = a\cos\varphi,$$
$$|AB_3| = |B_1B_2| = |B_1B|\cos\alpha = a\sin\varphi\cos\alpha,$$
$$|B_3B_4| = |B_3B_2|\cos\alpha = a\cos\varphi\cos\alpha,$$

and, finally,

$$|AB_4| = \sqrt{|AB_3|^2 + |B_3B_4|^2}$$
$$= a\sqrt{\sin^2\varphi\cos^2\alpha + \cos^2\varphi\cos^2\alpha} = a\cos\alpha.$$

Hence it follows that the length of any line segment lying in the plane of a lateral face after a twofold projection indicated in the conditions of the problem will be multiplied by $\cos\alpha$ (with the aid of translation we bring one of the end points of the given line segment into the vertex A). Consequently, in such projecting any figure will go into the figure similar to it with the ratio of similitude equal to $\cos\alpha$.

194. The statement of the problem follows from the equalities

$$V_{AA_1BC} = V_{AA_1B_1C} = V_{AA_1B_1C_1}$$

and similar equalities for the volumes of the pyramids AA_1CD and AA_1DB.

195. Let M denote the point of intersection of the straight lines CB_1 and C_1B. The vertex A lies on DM.

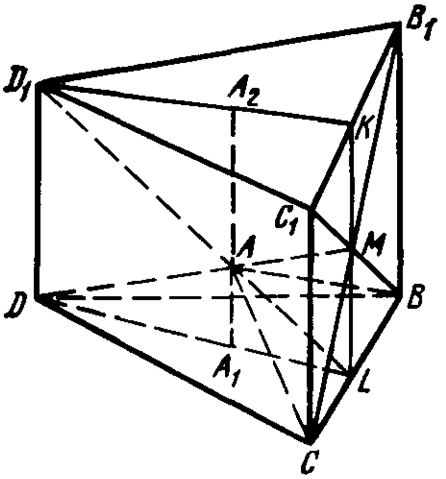


Fig. 43

Through the points D, D_1, and A pass a plane. Denote by K and L the points of its intersection with C_1B_1 and CB, and by A_2 the point of intersection of the line AA_1 with D_1K (Fig. 43). From the fact that CC_1B_1B is a trapezoid and KL passes through the point of intersection of its diagonals it follows that $|KM| = |ML|$. Further, considering the trapezoid D_1KLD, we will prove that $|AA_1| = \frac{1}{2}|AA_2|$. Consequently,

$$V_{ABCD} = \frac{1}{3} V_{A_2BCD}.$$

But it follows from the preceding problem that $V_{A_2BCD} = V_{A_1B_1C_1D_1}$. Thus, the ratio of the volumes of the pyramids $A_1B_1C_1D_1$ and $ABCD$ is equal to 3.

196. Introduce the following notation: $ABCD$ is the given tetrahedron $|BC| = a$, $|CA| = b$, $|AB| = c$, $|DA| = m$, $|DB| = n$, $|DC| = p$. Let then G denote the centre of gravity of the triangle ABC, N the point of intersection of the straight line DM with the circumscribed sphere, and K the point of intersection of the

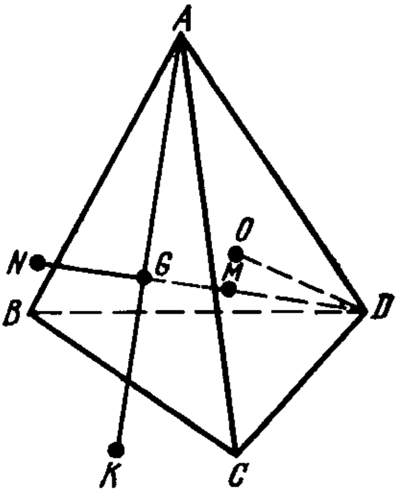


Fig. 44

straight line AG with the circle circumscribed about the triangle ABC (Fig. 44). Let us take advantage of the following equality which is readily proved:

$$|AG|\cdot|GK| = \frac{1}{9}(a^2+b^2+c^2).$$

Then

$$|DG|\cdot|GN| = |AG|\cdot|GK| = \frac{1}{9}(a^2+b^2+c^2),$$

consequently,

$$|GN| = \frac{a^2+b^2+c^2}{9t},$$

where

$$t = |DG| = \frac{1}{3}\sqrt{3m^2+3n^2+3p^2-a^2-b^2-c^2} \qquad (1)$$

(see Problem 51), $|DN| = |DG| + |GN| = t + \frac{a^2+b^2+c^2}{9t} = \frac{m^2+n^2+p^2}{3t}$. The assertion that OM is perpendicular to DM, is equivalent to the assertion that $|DN| = 2|DM| = 2 \cdot \frac{3}{4}|DG| = \frac{3}{2}t$, that is, $\frac{m^2+n^2+p^2}{3t} = \frac{3}{2}t$, whence replacing t by its expression (1), we get

$$a^2 + b^2 + c^2 = m^2 + n^2 + p^2. \tag{2}$$

If A_1, B_1, C_1 are the centres of gravity of the respective faces DBC, DCA, and DAB, then in the tetrahedron $A_1B_1C_1D$ we will have

$$|B_1C_1| = \frac{a}{3}, \quad |C_1A_1| = \frac{b}{3}, \quad |A_1B_1| = \frac{c}{3},$$

$$|DA_1| = \frac{2}{3}m_a, \quad |DB_1| = \frac{2}{3}n_b, \quad |DC_1| = \frac{2}{3}p_c,$$

where m_a, n_b, and p_c are the respective medians to the sides BC, CA, and AB in the triangles DBC, DCA, and DAB. If now t_1 is the distance from the vertex D to the point M, then, since M, by the hypothesis, lies on the surface of the sphere circumscribed about the tetrahedron $A_1B_1C_1D$ and the line DM passes through the centre of gravity of the triangle $A_1B_1C_1$, to determine the quantity $|DM|$ we may take advantage of the formula obtained above for $|DN|$, that is,

$$|DM| = \frac{4m_a^2+4n_b^2+4b_c^2}{27t_1},$$

where

$$t_1 = \frac{1}{9}\sqrt{12(m_a^2+n_b^2+p_c^2)-a^2-b^2-c^2}.$$

Taking advantage of the formula for the length of the median of a triangle, we get

$$|DM| = \frac{4m^2+4n^2+4p^2-a^2-b^2-c^2}{27t_1},$$

where

$$t_1=\frac{2}{9}\sqrt{3m^2+3n^2+3p^2-a^2-b^2-c^2}=\frac{2}{3}t.$$

On the other hand, $|DM|=\frac{3}{4}t$, that is,

$$\frac{4m^2+4n^2+4p^2-a^2-b^2-c^2}{18t}=\frac{3}{4}t.$$

Replacing t by its expression (Formula (1)), we get (2) which was required to be proved.

197. Fix some axis of symmetry l. Then, if l' is also an axis of symmetry and l' does not intersect with l or intersect l but not at right angles, then the line l'', which is symmetric to l' with respect to l is also an axis of symmetry. This is obvious. And if some line l_1 is an axis of symmetry and intersects with, and is perpendicular to, l, then the line l_2 passing through the point of intersection of l and l_1 and perpendicular to them will also be an axis of symmetry. It is possible to verify it, for instance, in the following way. Let us take the lines l, l_1, and l_2 for the coordinate axes.

Applying, in succession, to the point $M(x, y, z)$ symmetry transformations with respect to the lines l and l_1, we will bring the point M first to the position $M_1(x, -y, -z)$, and then M_1 to $M_2(-x, -y, z)$. Thus, a successive application of symmetry transformations with respect to the lines l and l_1 is equivalent to symmetry with respect to l_2.

Our reasoning implies that all axes of symmetry, except for l, can be divided in pairs, that is, the number of symmetry axes is necessarily odd if it is finite.

198. Let M denote the projection of B on AD. Obviously, M belongs to the surface of the sphere with diameter AB. On the other hand, we can show that $|AM|\times|AD|=|AB|^2$. Hence it follows that all points M must belong to a certain spherical surface containing the given circle. Hence, points M belong to one circle along which these two spherical surfaces intersect.

199. Prove that the projections of the point M on the sides of the quadrilateral $ABCD$ lie on one and the same circle (if K and L are projections of M on AB and BC,

then the points B, K, M, and L lie in one circle, and, hence, $\widehat{MLK} = \widehat{MBK}$, $\widehat{MKL} = \widehat{MBL}$. The same for other sides).

Then take advantage of the result of Problem 198.

200. Since the centre of gravity lies on the lines joining the midpoints of the edges AB and CD, it will follow from the hypothesis that this line will be perpendicular to the edges AB and CD.

201. Let K and M denote the midpoints of the edges AB and CD. It follows from the hypothesis that the line KM passes through the point O which is the centre of the inscribed sphere; O is equidistant from the faces ACD and BCD. Consequently, the point K is also equidistant from these faces. Hence it follows that these faces are equivalent. In the same way, the faces ABC and ABD turn out to be equivalent. If we now project the tetrahedron on the plane parallel to the edges AB and CD, then its projection will be a parallelogram with diagonals AB and CD. Hence there follows the statement of our problem.

202. Rotate the cube through some angle about the diagonal AC_1. Since the plane of the triangle A_1BD is perpendicular to AC_1 and its sides are tangent to the ball inscribed in the cube, the sides of the triangle obtained from A_1BD after the rotation will also touch the inscribed ball. With the angle of rotation appropriately chosen, the face AA_1B_1B will go into the given plane, and the line segment MN will be a line segment of the rotated face.

203. Denote by α, β, γ the angles formed by rectangular faces with the fourth face. If S_1, S_2, S_3, S_4 are the respective areas of the faces, then $S_1 = S_4 \cos\alpha$, $S_2 = S_4 \cos\beta$, $S_3 = S_4 \cos\gamma$. After this, we may take advantage of the fact that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. This follows, for example, from the fact that the angles made by the altitude dropped on the fourth face with the lateral edges of the pyramid are also equal to α, β, and γ (see Problem 10).

204. Take a straight line perpendicular to the given plane and denote by α, β, and γ the angles made by this line with the edges of the cube. The projections of the edges on the plane take on the values $\sin\alpha$, $\sin\beta$, $\sin\gamma$. And since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, the sum of the

squares of the projections will be equal to

$$4a^2 (\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma) = 8a^2,$$

where a is the edge of the cube.

205. Through each edge of the tetrahedron pass a plane parallel to the opposite edge. We will obtain a cube with a tetrahedron inscribed in it. If the edge of the tetrahedron is b, then the edge of the cube will be equal to $b/\sqrt{2}$. The projection of each face of the cube is a parallelogram whose diagonals are equal to the projections of the edges of the tetrahedron. The sum of the squares of all diagonals is equal to the doubled sum of the squares of the projections of the edges of the tetrahedron and is equal to twice the sum of the squares of the projections of the edges of the cube.

Taking advantage of the result of the preceding problem, we get that the sum of the squares of the projections of the edges of a regular tetrahedron on an arbitrary plane is equal to $8 \frac{b^2}{2} = 4b^2$.

206. Consider first the case when the given straight lines are skew lines. Denote by A and B the positions of the points at some instant of time, k is the ratio of their velocities (the velocity of the body situated at the point A is k times the velocity of the other body). M and N are two points on the line AB such that $|AM| : |MB| = |AN| : |NB| = k$ (M is on the line segment AB), O is the midpoint of MN. The proof of the statement of our problem is divided into the following items:

(1) The points M, N, and O move in straight lines, the straight lines in which the points A, B, M, N, and O move are parallel to one plane.

(2) The lines in which the points M and N move are mutually perpendicular.

(3) If two straight lines are mutually perpendicular and represent skew lines, then any sphere constructed on the line segment whose end points lie on these lines, as on the diameter, passes through the points P and Q, where PQ is a common perpendicular to these lines (P and Q are situated on the straight lines).

(4) The locus of points L such that $|AL| : |LB| = k$ is the surface of the sphere constructed on MN, as on the diameter.

From the statements (1) to (4) it follows that the circle whose existence is asserted in the problem is the circle obtained by rotating the point P (or Q) about a straight line in which the point O moves, where P and Q are the end points of the common perpendicular to the straight lines in which the points M and N are displaced.

Items (1) and (2) can be proved, for instance, in the following way. Let A_0 and B_0 denote the positions of the points at a certain fixed instant of time. Let us project our points parallel to the straight line A_0B_0 on a plane parallel to the given lines. The points A_0 and B_0 will be projected into one point C, and the points A, B, M, N, and O will be projected into the respective points A', B', M', N', and O'. Then the points M' and N' will represent the end points of the bisectors of the interior and the exterior angle C of the triangle $A'B'C'$. Hence, M', N', and O' move in straight lines, and $\widehat{M'CN'} = 90°$. Hence it follows that the points M, N, and O also displace in straight lines, since it is obvious that each of these points lies in the fixed plane parallel to the given lines. Item (3) is obvious. Item (4) follows from the corresponding statement of plane geometry.

In the case when the points A and B move in two intersecting lines, the relevant reasoning is somewhat changed. The problem is reduced to the proof that in the plane containing the given lines there are two fixed points P and Q such that $|AP|:|PB| = |AQ|:|QB| = k$.

207. Let O denote the centre of the ball, r its radius, AP and BQ the tangents to the ball (P and Q being the points of tangency), M the point of intersection of the lines AP and BQ. Setting $|OA| = a$, $|OB| = b$, $|PM| = |QM| = x$. Then $|OM|^2 = r^2 + x^2$, $|AM|^2 = (\sqrt{a^2 - r^2} \pm x)^2$, $|BM|^2 = (\sqrt{b^2 - r^2} \pm x)^2$.

If the signs are of the same sense, then the following relationship is fulfilled:

$$\sqrt{b^2-r^2}\,|AM|^2 - \sqrt{a^2-r^2}\,|BM|^2$$

$$+(\sqrt{a^2-r^2}-\sqrt{b^2-r^2})\,|OM|^2 = l_1. \qquad (1)$$

If the signs are opposite, then

$$\sqrt{b^2-r^2}\,|AM|^2+\sqrt{a^2-r^2}\,|BM|^2 -\left(\sqrt{a^2-r^2}+\sqrt{b^2-r^2}\right)|OM|^2=l_2, \quad (2)$$

where l_1 and l_2 are constants depending on r, a, and b.

Since the sum of the coefficients of $|AM|^2$, $|BM|^2$ and $|OM|^2$ in Equations (1) and (2) is equal to zero, the locus of points M for which one of these relationships is fulfilled is a plane. In both cases this plane is perpendicular to the plane OAB.

208. Let ABC be the given triangle whose sides, as usually, are equal to a, b, and c. The radii of the three balls touching one another and the plane of the triangle at points A, B, and C are respectively equal to $\frac{bc}{2a}$, $\frac{ca}{2b}$, $\frac{ab}{2c}$. Denote by x the radius of the ball touching the three given balls and the plane of the triangle, M is the point of tangency of this ball and the plane. We have:

$$|MA|=2\sqrt{\frac{bcx}{2a}}, \quad |MB|=2\sqrt{\frac{acx}{2b}},$$

$$|MC|=2\sqrt{\frac{abx}{2c}}.$$

Consequently, $|MA|:|MB|=b:a$, $|MB|:|MC|=c:b$ or $|MA|:|MB|:|MC|=bc:ac:ab$.

For any irregular triangle there are exactly two points M_1 and M_2 for which this relationship is fulfilled. Here we take advantage of Bretschneider's theorem. Let $ABCD$ be an arbitrary plane quadrilateral. Let $AB=a$, $BC=b$, $CD=c$, and $DA=d$, $AC=m$ and $BD=n$. The sum of the angles $\hat{A}+\hat{C}=\varphi$. Then the equality $m^2n^2=a^2c^2+b^2d^2-2abcd\cos\varphi$ holds. We then obtain that if $\hat{A}=\alpha$ is the smallest angle of the triangle, then the angles $\widehat{BM_1C}$ and $\widehat{BM_2C}$ are equal to $60°+\alpha$ and $60°-\alpha$.

Let $\widehat{BM_1C} = 60° + \alpha$. Write for the triangle BM_1C the theorem of cosines, denoting the radius of the ball touching the plane at point M_1 by r $(x = r)$,

$$a^2 = \frac{2acr}{b} + \frac{2abr}{c} - 4ar\cos(60° + \alpha)$$

$$\Rightarrow \frac{1}{r} = 2\left(\frac{c}{ab} + \frac{b}{ac} - \frac{2\cos(60° + \alpha)}{a}\right). \qquad (1)$$

Analogously, designating the radius of the ball touching the plane at point M_2 by ρ, we get

$$\frac{1}{\rho} = 2\left(\frac{c}{ab} + \frac{b}{ac} - \frac{2\cos(60° - \alpha)}{a}\right). \qquad (2)$$

Subtracting (2) from (1), we obtain

$$\frac{1}{r} - \frac{1}{\rho} = \frac{4[\cos(60° - \alpha) - \cos(60° + \alpha)]}{a}$$

$$= \frac{8\sin 60° \sin\alpha}{a} = \frac{2\sqrt{3}}{R},$$

which was required to be proved.

209. Let M denote the midpoint of AB, O_1 and O_2 the centres of the balls, R_1 and R_2 their radii, then

$$|MO_1|^2 - |MO_2|^2 = \left(R_1^2 + \frac{|AB|^2}{4}\right) - \left(R_2^2 + \frac{|AB|^2}{4}\right)$$
$$= R_1^2 - R_2^2.$$

This means that the midpoints of all the line segments of common tangents to the given balls lie in one and the same plane which is perpendicular to the line segment O_1O_2. Hence follows the truth of the statement of our problem.

210. Such pentagon does not exist.

211. Let $A_1A_2A_3A_4A_5$ be the given pentagon. It follows from the hypothesis that all the diagonals of the pentagon are equal to one another. Choose three vertices of the pentagon so that the remaining two vertices lie on one side of the plane determined by the three chosen vertices, say, A_2, A_3, and A_5. Then the vertices A_1 and A_4 will be symmetric to each other with respect to the plane

passing through the midpoint of A_2A_3 perpendicular to A_2A_3. This follows from the fact that the triangle $A_2A_3A_5$ is isosceles, $|A_2A_5| = |A_3A_5|$, A_1 and A_4 lie on one side of the plane $A_2A_3A_5$, and $|A_1A_2| = |A_4A_3|$, $|A_1A_5| = |A_4A_5|$, and $|A_1A_3| = |A_4A_2|$. Hence, the points A_1, A_2, A_3, and A_4 lie in one plane. The further reasoning is clear. The cases when the sought-for plane passes through other vertices are considered in a similar way.

212. Let M denote the point of intersection of the diagonal AC_1 and the plane A_1BD. Then M is the point

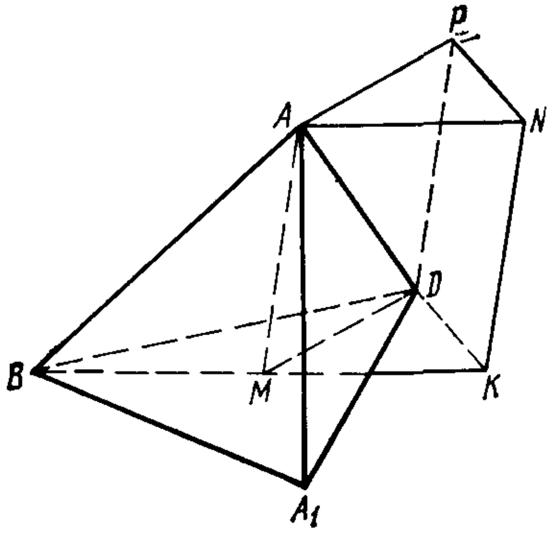


Fig. 45

of intersection of the medians of the triangle A_1BD (so-called median point) and, besides, M divides the diagonal AC_1 in the ratio 1 : 2, that is $|AM| = \frac{1}{3}d$.

Consider the pyramid ABA_1D (Fig. 45). On the line BM take a point K such that $|MK| = |BM|$, and construct the prism $MKDANP$. You can easily notice that the distances between the lateral edges of this prism are equal to the respective distances from the points A_1, B, and D to AM. Consequently, the sides of the section perpendicular to the lateral edges of the prism $MKDANP$ are equal to these distances. Further, the volume of the pyramid ABA_1D is equal to the volume of the constructed prism and amounts to one sixth the volume of the parallelepiped, i.e. $\frac{1}{6}V = \frac{1}{3}dS$, $V = 2dS$.

213. Let M denote the centre of gravity of the tetrahedron $ABCD$. The volume of the pyramid $MABC$ is one fourth the volume of the given tetrahedron. Complete the pyramid $MABC$ to get a parallelepiped so that the line segments MA, MB, MC are its edges. Figure 46 repre-

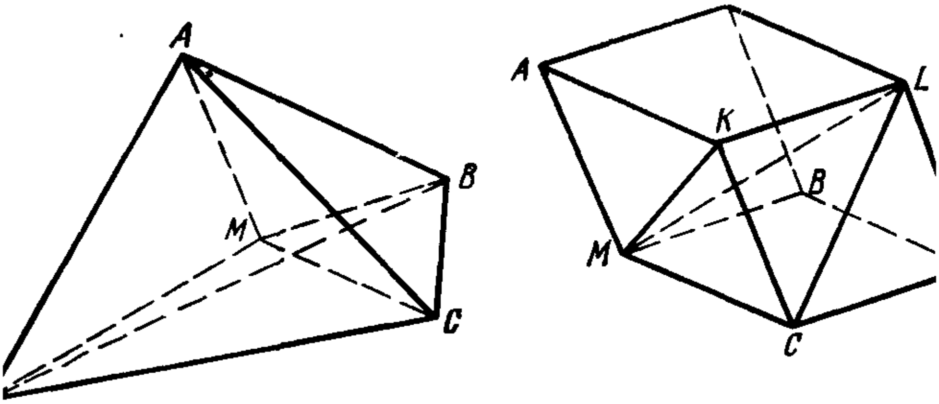


Fig. 46

sents this parallelepiped separately. It is obvious that the edges MC, CK, KL and diagonal ML of this parallelepiped are respectively equal and parallel to MC, MA, MB, and MD. But the volumes of the pyramids $MABC$ and $MCKL$ are equal to each other, that is, each of them is equal to $\frac{1}{4}V_{ABCD}$. Consequently, the volume of the tetrahedron in question equals $\left(\frac{4}{3}\right)^2 \cdot \frac{1}{4} V_{ABCD} = \frac{16}{27} V$.

214. When solving Problem 180, we proved that the sum of the vectors, perpendicular to the faces of the tetrahedron, directed towards outer side with respect to the tetrahedron, and whose lengths are numerically equal to the area of the corresponding faces, is equal to zero. Hence follows the existence of the tetrahedron $KLMN$.

In finding the volume of the tetrahedron, we shall take advantage of the following formula:

$$V = \frac{1}{6} abc \sin\alpha \sin\beta \sin C,$$

where a, b, and c denote the respective lengths of the edges emanating from a certain vertex of the tetrahedron, α and β two plane angles at this vertex, and C the dihedral angle between the planes of the faces corresponding to the angles α and β. If now α, β, and γ are all plane angles at this vertex and A, B, and C are dihedral angles, then

$$V^3 = \left(\frac{1}{6}\right)^3 a^3b^3c^3 \sin^2\alpha \sin^2\beta \sin^2\gamma \sin A \sin B \sin C. \quad (1)$$

Take now a point inside the tetrahedron, and from it drop perpendiculars on the three faces of the tetrahedron corresponding to the trihedral angle under consideration, and on each of them lay off line segments whose lengths are numerically equal to the areas of these faces. Obviously, the volume of the tetrahedron formed by these line segments is equal to that of the tetrahedron $KLMN$. The plane angles at the vertex of the trihedral angle formed by these line segments are equal to $180° - A$, $180° - B$, $180° - C$, and the dihedral angles to $180° - \alpha$, $180° - \beta$, $180° - \gamma$. Consequently, making use of Equality (1), we get for the volume W of this tetrahedron

$$W^3 = \left(\frac{1}{6}\right)^3 S_1^3 S_2^3 S_3^3 \sin^2 A \sin^2 B \sin^2 C \sin\alpha \sin\beta \sin\gamma, \quad (2)$$

where S_1, S_2, S_3 are the areas of the faces formed by the edges a, b, and c, respectively, that is, $S_1 = \frac{1}{2}\, ab \sin\gamma$, $S_2 = \frac{1}{2} bc \sin\alpha$, $S_3 = \frac{1}{2} ca \sin\beta$.

Replacing S_1, S_2, S_3 in (2), we get

$$W^3 = \left(\frac{1}{6}\right)^3 \left(\frac{1}{2}\right)^9 a^6b^6c^6 \sin^4\alpha \sin^4\beta \sin^4\gamma \sin^2 A \sin^2 B \times \sin^2 C. \quad (3)$$

Comparing Equations (1) and (3), we obtain

$$W = \frac{3}{4} V^2.$$

215. The statement of the problem follows from the fact that the products of the line segments into which each of these chords is divided by the point of intersection are equal.

217. The statement of our problem follows from the following fact of plane geometry. If through a point P lying outside of the given circle two straight lines are drawn intersecting the circle at the respective points A and A_1, B and B_1, then the line A_1B_1 is parallel to the circle circumscribed about PAB passed through the point P.

Thus, the set of points under consideration will belong to the plane parallel to the plane which touches (at the point P) the sphere passing through the given circle and point P.

218. The equation

$$(x - a)^2 + (y - b)^2 = k^2 (z - c)^2$$

describes a conical surface whose vertex is found at the point S (a, b, c), the axis is parallel to the z-axis, $k = \tan \alpha$, where α is the angle between the axis of the cone and its generatrix. Subtracting from each other the equations of two conical surfaces with axes parallel to the z-axis, equal parameters k, but different vertices, we get a linear dependence relating x, y, and z.

219. Denote by F the point of intersection of the lines KL and MN and by E the point of intersection of the line PF and the sphere passing through the points P, A, B, and C (supposing that P does not lie in the plane of the face ABC).

The points P, Q, R, and E belong to one circle representing the section of the sphere passing through the points P, A, B, and C by the plane passing through the points P, K, and L. But since F is the point of intersection of the lines KL and MN, the points P, S, T, and E must belong to the circle which is the section of the sphere passing through the points P, A, C, and D by the plane determined by the points P, M, and N. Consequently, the points P, Q, R, S, and T lie on two circles having two common points P and E, and such two circles belong to one sphere.

Remark. We have considered the case of the general position of the given points. To get a complete solution we have to consider several particular cases, say, P lies in the plane of the face, KL and MN are parallel lines, and so on.

220. Let the edges SA, SB, SC, and SD of a quadrihedral angle be elements of a cone whose axis is SO. Then in the trihedral angle formed by the lines SO, SB, and SC, the dihedral angles with the edges SA and SB are equal. Considering three other such angles, we get easily that the sums of opposite dihedral angles of the given quadrihedral angle are equal.

Conversely. Let the sums of opposite dihedral angles be equal. Consider the cone with the lines SA, SB, and SC as its elements. Suppose that SD is not an element. Denote by SD_1 the straight line along which the surface of the cone and the plane ASD intersect. We will obtain two quadrihedral angles $SABCD$ and $SABCD_1$ in each of which the sums of opposite dihedral angles are equal. This will imply that in the trihedral angle which is complementary to the angle $SCDD_1$ (see the solution of Problems 165 and 166) one plane angle is equal to the sum of two others which is impossible.

221. Let all the vertices of the hexahedron $ABCDEFKL$, except for C, lie on the surface of the

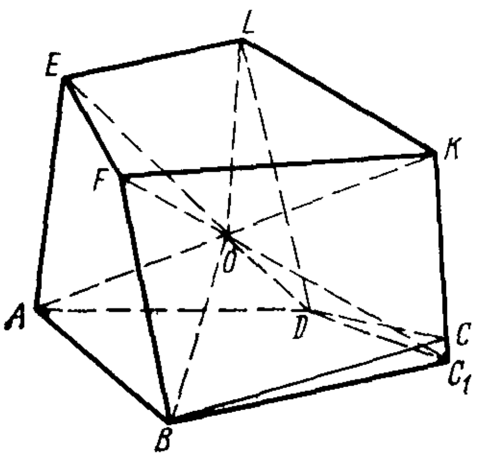


Fig. 47

sphere with centre O (Fig. 47). Denote by C_1 the point of intersection of the line KC with the surface of the sphere.

For the sake of brevity we shall symbolize by $\sphericalangle FEL$ the dihedral angle between the planes FEO and FLO (the remaining dihedral angles are denoted in a similar way). Using the direct statement of Problem 220, we may write:

$$\measuredangle FEL + \measuredangle FKL = \measuredangle EFK + \measuredangle ELK,$$
$$\measuredangle AEF + \measuredangle ABF = \measuredangle EAB + \measuredangle EFB,$$
$$\measuredangle AEL + \measuredangle ADL = \measuredangle ELD + \measuredangle EAD,$$
$$\measuredangle FKC_1 + \measuredangle FBC_1 = \measuredangle KFB + \measuredangle KC_1B,$$
$$\measuredangle LKC_1 + \measuredangle LDC_1 = \measuredangle KLD + \measuredangle KC_1D.$$

Adding together all these equalities and taking into consideration that the sum of any three dihedral angles having a common edge (say, OE) is equal to 2π, we get

$$\measuredangle ABC_1 + \measuredangle ADC_1 = \measuredangle BAD + \measuredangle BC_1D,$$

and this means (see the converse statement of Problem 220) that the edges OA, OB, OC_1, and OD are elements of one cone. Hence it follows that C_1 lies in the plane ABD, that is, C_1 coincides with C.

The case when O is situated outside the polyhedron requires a separate consideration.

222. Let $ABCD$ be the given tetrahedron, K, L, M, N, P, and Q the given points on the respective edges AB, AC, AD, BC, CD, and DB. Denote by D_1 the point of intersection of the circles passing through K, B, N and C, L, N. It is not difficult to prove that the point D_1 belongs to the circle passing through the points A, K, and L. Analogously, we determine the points A_1, B_1, and C_1 in the planes BCD, ACD, and ADB. Let, finally, F be the point of intersection of the three spheres circumscribed about the tetrahedrons $KBNQ$, $LCNP$, and $NDPQ$. Take advantage of the result of Problem 221. In the polyhedron with vertices B, N, A_1, Q, K, D_1, F, C_1 all the vertices lie on the surface of the sphere, five faces BKD_1N, BKC_1Q, BNA_1Q, D_1NA_1F, A_1QC_1F are plane quadrilaterals, consequently, KD_1FC_1 is also a plane quadrilateral. In the same manner, prove that LD_1FB_1 and MB_1FC_1 are also plane quadrilaterals.

And, finally, in the hexahedron $AKD_1LMB_1FC_1$ seven vertices A, K, D_1, L, M, B_1, C_1 lie on the surface of the sphere passing through A, K, L, and M, hence, the point F also lies on the same sphere.

Section 3

224. Let S be the vertex of the angle. Cut the angle by a plane so as to form a pyramid $SABCD$ in which $ABCD$ is the base and the opposite lateral edges are equal:

$|SA| = |SC|, \quad |SB| = |SD|.$

(Prove that this can be done always.) Since the plane angles at vertices are equal, $ABCD$ is a rhombus. Let O be the point of intersection of AC and BD. Set $|AC| = 2x$, $|BD| = 2y$, $|SO| = z$ and suppose that $x \leqslant y$. If $\widehat{ASC}$ and $\widehat{BSD}$ are acute, then $z > y$, and this means that in the triangle ASB $|AB| < |AS| < |BS|$, that is, $\widehat{ASB}$ is the smallest angle of this triangle, $\widehat{ASB} < 60°$.

The supposition that both angles are obtuse is considered in the same manner.

225. From Sh to $\frac{4}{3}$ Sh.

226. The greatest volume is possessed by the tetrahedron two opposite edges of which are mutually perpendicular and are the diameters of the bases. Its volume is equal to $\frac{2}{3} R^2h$.

227. Let $|AB| = |BC| = 1$, $|AA_1| = x$.

$$V_{DD_1BC_1} = \frac{1}{3} S_{DBD_1} \cdot \frac{\sqrt{2}}{2} = \frac{1}{6} x.$$

On the other hand,

$$V_{DD_1BC_1} = \frac{1}{3} S_{DBC_1} |D_1B| \sin\varphi$$
$$= \frac{\sqrt{2}}{6} \cdot \sqrt{\frac{1}{2} + x^2} \cdot \sqrt{2+x^2} \sin\varphi,$$

where φ is the angle between D_1B and the plane DBC_1.

Thus,

$$\sin\varphi = \frac{x}{\sqrt{(2+x^2)(1+2x^2)}}, \quad \frac{1}{\sin^2\varphi} = 2x^2 + \frac{2}{x^2} + 5 \geqslant 9,$$

whence it follows that the greatest value of φ will be $\arcsin\frac{1}{3}$.

228. Let the altitude of the prism be equal to 1, $|AM| = x$. Circumscribe a circle about the triangle $|A_1MC_1|$. Consider the solid obtained by revolving the arc A_1MC_1 of this circle about the chord A_1C_1. The angle A_1MC_1 will be the greatest if the line AB touches the surface of the solid thus generated. The latter happens if the lines MO and AB, where O is the centre of the circle circumscribed about the triangle ABC, are mutually perpendicular; hence, the line MO divides A_1C_1 in the ratio $\frac{|AM|}{|MB|}=\frac{x}{2-x}$.

On the other hand, it is possible to show that MO divides A_1C_1 in the ratio $\frac{|A_1M|\cos\widehat{A_1C_1M}}{|C_1M|\cos\widehat{C_1A_1M}}$. Expressing the sides and cosines of the angles of the triangle A_1MC_1 in terms of x, we get the equation

$$\frac{(1+x^2)(4-x)}{x(9-4x+x^2)}=\frac{x}{2-x} \Leftrightarrow x^2+3x-4=0,$$

whence $x=1$. The greatest value of the angle A_1MC_1 equals $\frac{\pi}{2}$.

229. The lines AE and CF are mutually perpendicular. Let Q_1 be the projection of Q on the plane ABB_1A_1. Q_1 lies on the line segment BL, where L is the midpoint of AA_1. Let N be the point of intersection of AE and LB. It is easy to find that $|AN|=\frac{1}{\sqrt{5}}$. Setting $|AP|=\frac{1}{\sqrt{5}}+x$, $|NQ_1|=y$, we get $|PM|^2=\frac{8}{5}+\left(\frac{1}{\sqrt{5}}+x\right)^2$, $|PQ|^2=x^2+y^2+1$, $\frac{|PM|^2}{|PQ|^2}$ attains the greatest value for $y=0$. It remains to find the

greatest value of the fraction $\frac{9/5+(2/\sqrt{5})\,x+x^2}{x^2+1}$.

This value is attained for $x=\frac{1}{\sqrt{5}}$.

Answer: $\sqrt{2}$.

230. Consider the triangle KLM representing the projection of the given triangle on the plane $ABCD$, K lying on the line CB, L on CD, M on CA. If $|CK|=x$, then $|CL|=|a-x|$, $|CM|=\sqrt{2}\left|a-\frac{x}{2}\right|$.

It is rather easy to get that

$$S_{KLM}=\frac{1}{2}\left|x\,(a-x)-a\left(a-\frac{x}{2}\right)\right|$$
$$=\frac{1}{4}\,(2x^2-3ax+2a^2).$$

The least value is equal to $\frac{7a^2}{32}$.

231. Let x denote the altitude of the parallelepiped. Consider the section of the pyramid by the plane passing at a distance x from its base. The section represents a square with side $(1-x)$; a rectangle of area s which is a face of the parallelepiped is inscribed in the square. Two cases are possible:

(1) The base of the parallelepiped is a square with side $\sqrt{s}$. The diagonal of the parallelepiped $d=\sqrt{x^2+2s}$, and

$$(1-x)\,\frac{\sqrt{2}}{2}\leqslant\sqrt{s}\leqslant(1-x)$$

or

$$1-\sqrt{2s}\leqslant x\leqslant 1-\sqrt{s}.$$

Thus, in this case if $s<\frac{1}{2}$, $1-2\sqrt{2s}+4s\leqslant d^2\leqslant 1-2\sqrt{s}+3s$, and if $s\geqslant\frac{1}{2}$, $2s<d^2\leqslant 1-2\sqrt{s}+3s$.

(2) The sides of the face of the parallelepiped inscribed in the section are parallel to the diagonals of the section. Let us denote them by y and z. Our problem consists in investigating the change of the function $d^2 = x^2 + y^2 + z^2$ under the conditions

$$\begin{cases} yz = s, \\ y + z = (1 - x)\sqrt{2}. \end{cases}$$

(The latter system is consistent if $1 - x \geqslant \sqrt{2s}$, $0 < x \leqslant 1 - \sqrt{2s}$.) We have

$$d^2 = x^2 + (y + z)^2 - 2yz = x^2 + 2(1 - x)^2 - 2s$$
$$= 3x^2 - 4x + 2 - 2s.$$

If $s \leqslant \frac{1}{18}$, then the least value of d^2 is attained for $x = \frac{2}{3}$, and if $s > \frac{1}{18}$, then for $x = 1 - \sqrt{2s}$. Besides, $d^2 < 2 - 2s$. Combining the results of items (1) and (2), we get the answer.

Answer: if $0 < s \leqslant \frac{1}{18}$, then

$$\sqrt{\frac{2}{3} - 2s} \leqslant d < \sqrt{2 - 2s};$$

if $\frac{1}{18} < s < \frac{7 + 2\sqrt{6}}{25}$, then

$$\sqrt{1 - 2\sqrt{2s} + 4s} \leqslant d < \sqrt{2 - 2s};$$

if $\frac{7 + 2\sqrt{6}}{25} \leqslant s < \frac{1}{2}$, then

$$\sqrt{1 - 2\sqrt{2s} + 4s} \leqslant d \leqslant \sqrt{1 - 2\sqrt{s} + 3s};$$

if $\frac{1}{2} \leqslant s < 1$, then

$$\sqrt{2s} < d \leqslant \sqrt{1 - 2\sqrt{s} + 3s},$$

232. Cut the polyhedron $ABCA_1MNC_1$ by the plane passing at a distance h from the plane $A_1B_1C_1$ and project the section thus obtained on the plane $A_1B_1C_1$ (Fig. 48). In the figure, the projection of this section is

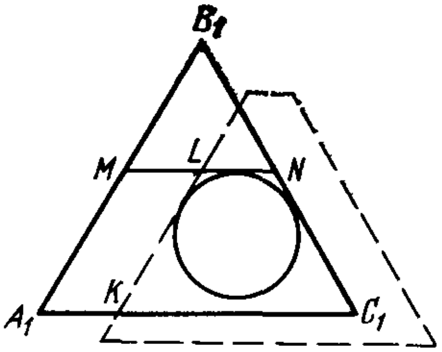


Fig. 48

shown in dashed line. It is obvious that the circle of the base of the cylinder must be located inside the trapezoid $KLNC_1$ (K, L are the respective points of intersection of A_1C_1 and MN with the projection of this section). If $h = 3$, then the section plane coincides with the plane ABC and the points K and L with the midpoints of the sides B_1C_1 and A_1C_1. If $h < 3$, $|ML| = |A_1K| = \frac{h}{3}$, $|LN| = 1 - \frac{h}{3}$, $|KC_1| = 2 - \frac{h}{3}$.

We can readily verify that for $h \leqslant \frac{3}{2}$ the radius of the greatest circle contained in the trapezoid $KLNC_1$ is equal to $\frac{\sqrt{3}}{4}$, and for $h > \frac{3}{2}$ this radius is equal to the radius of the circle inscribed in a regular triangle with side $|KC| = 2 - \frac{h}{3}$, that is, it is equal to $\left(2 - \frac{h}{3}\right)\frac{\sqrt{3}}{6}$.

Answer: (a) if $0<h\leqslant\frac{3}{2}$, $V=\frac{3}{16}\pi h$; if $\frac{3}{2}<h\leqslant 3$, $V=\frac{\pi}{12}h\left(2-\frac{h}{3}\right)^2$;

(b) the greatest value of the volume will be obtained for $h=2$, $V=\frac{8\pi}{27}$.

233. If the plane passed through our line segment parallel to the face ABB_1A_1 cuts CB at the point K so that $|CK|=x$, then the projection of the line segment on the face ABC has a length x, and its projection on the edge CC_1 is equal to $|a-2x|$; thus, the length of the line segment will be equal to

$$\sqrt{x^2+(a-2x)^2}=\sqrt{5x^2-4ax+a^2}.$$

The minimal length is equal to $\frac{a}{\sqrt{5}}$.

234. The following statement is an analogue of our problem in the plane. Given an angle and a point N inside it. Consider all possible triangles formed by the sides of the angle and straight line passing through the point N. Among such triangles, the smallest area is possessed by the one for which the side passing through N is bisected by the point N.

Let us return to our problem. Let M be the given point inside the trihedral angle. The plane passing through the point M intersects the edges of the trihedral angle at points A, B, and C. Let the line AM intersect BC at N. Then, if the passed plane cuts off a tetrahedron of the least volume, the point N must be the midpoint of BC. Otherwise, rotating the plane about the line AN, we will be able to reduce the volume of the tetrahedron.

235. If h is the altitude of the segment, then its volume is equal to $\frac{1}{2}Sh-\frac{1}{3}\pi h^3$. The greatest volume will be achieved for $h=\sqrt{\frac{S}{2\pi}}$; it will be equal to $\frac{S}{3}\sqrt{\frac{S}{2\pi}}$.

236. Note that the shadow thrown only by the upper face of the cube (assuming that all the remaining faces

are transparent) represents a square $\frac{ab}{b-a}$ on a side. Hence it follows that the area of the shadow cast by the cube will be the least when the source of light is located above the upper face (only the upper face of the cube is illuminated); it will be equal to $\left(\frac{ab}{b-a}\right)^2$ with the area of the lower face of the cube taken into account.

237. The statement (1) is true, let us prove this. Denote by P_1 the polygon obtained when our polygon is cut by a plane not passing through its centre, S denoting the area of this polygon. P_2 is a polygon symmetric to P_1 with respect to the centre of the polygon. Let us denote by Π the smallest convex polyhedron containing P_1 and P_2 (Π is called the *convex shell* of P_1 and P_2). Obviously, Π is a central-symmetric polygon, its centre coincides with the centre of the original polyhedron. All the vertices of Π are either vertices of P_1 or vertices of P_2. Let P denote the polygon obtained when Π is intersected by the plane passing through the centre parallel to the faces of P_1 and P_2, q its area. Let us take a face N of the polyhedron Π different from P_1 and P_2. It is obvious that any section of the polyhedron Π by a plane parallel to N must intersect either simultaneously all the three polygons P_1, P_2, and P or none of them. Since the polyhedron Π is convex, the line segments l_1, l_2, and l along which this plane cuts P_1, P_2, and P are related as follows: $l \geqslant \frac{1}{2}(l_1 + l_2)$. Hence it follows that $q \geqslant S$. (We integrate the inequality $l \geqslant \frac{1}{2}(l_1 + l_2)$ with respect to all possible planes parallel to N.)

The statement (2) is false. Let us construct an example. Consider in a rectangular Cartesian coordinate system the polyhedron whose points satisfy the inequality $|x| + |y| + |z| \leqslant 1$. (This polyhedron represents a regular octahedron.) All the faces of this polyhedron are regular triangles with side $\sqrt{2}$ and radius of the circumscribed circle $\sqrt{\frac{2}{3}}$. The section of this polyhedron by a plane passing through the origin and parallel to any face represents a regular hexagon with side $\frac{\sqrt{2}}{2}$ and the same radius of the circumscribed circle. But $\frac{\sqrt{2}}{2} < \sqrt{\frac{2}{3}}$.

Remark. For an arbitrary convex central-symmetric solid the following statement is true. Let R and R_0 denote the radii of the smallest circles containing the sections of the given solid by two parallel planes, the second plane passing through the centre; then $R_0 \geqslant \frac{\sqrt{3}}{2} R$. As we have already seen, an equality in this case is achieved for a regular octahedron.

238. 4/3.

239. Let A and B be the vertices of the cones, M and N two points on the circle of the bases, L a point diametrically opposite to the point M ($|AM| = \sqrt{r^2 + H^2}$, $|BM| = \sqrt{r^2 + h^2}$). Through M pass a plane perpendicular to AM and denote the projections of B, N, and L on this plane by B_1, N_1, and L_1. The distance between AM and BN is equal to the distance between M and B_1N_1, and cannot exceed $|MB_1|$.

The condition $h \leqslant H$ implies that $|MB_1| \leqslant |ML_1|$, that is, the point B_1 is situated inside, or on the boundary of, the projection of the base of the cones on the passed plane, and the distance between M and B_1N_1 is equal to MB_1 if MB_1 and B_1N_1 are mutually perpendicular.

Answer: $\frac{(h+H)r}{\sqrt{r^2+H^2}}$.

240. Extend the edge B_1B beyond the point B and on the extension take a point K such that $|BK| = a$. As is readily seen, K is equidistant from all the sides of the quadrilateral AB_1CD. On the diagonal B_1D take a point L such that $\frac{|B_1L|}{|LD|} = \sqrt{2}$. The point L is the end point of the bisectors of the triangles B_1AD and B_1CD and, hence, L is also equidistant from the sides of the quadrilateral AB_1CD. Now, we can prove that all the points of the line KL are equidistant from the sides

of the quadrilateral. Thus, the sought-for radius is equal to the shortest distance between the line KL and any of the lines forming the quadrilateral AB_1CD. Find the distance, say, between the lines KL and AD. Projecting the points K and L on the plane CDD_1C_1, we get the points K_1 and L_1. The desired distance is equal to the distance from the point D to the line K_1L_1.

Answer : $a\sqrt{1-\frac{\sqrt{2}}{2}}$.

241. Let the diagonal AC_1 lie on the edge of the dihedral angle, the faces of the angle intersect the edges of the cube at points M and N. It is not difficult to notice that if the volume of the part of the cube enclosed inside this angle reaches its greatest or smallest value, then the areas of the triangles AC_1M and AC_1N must be equal (otherwise, rotating the angle in the required direction, we shall be able both to increase and decrease this volume).

If $0 < \alpha \leqslant 60°$, then the part of the cube under consideration has a volume contained in the interval from $\frac{1}{2\sqrt{3}\cot\frac{\alpha}{2}}$ to $\frac{1}{3\left(1+\sqrt{3}\cot\frac{\alpha}{2}\right)}$. For $\alpha = 60°$ this volume is constant and is equal to 1/6.

For $60° < \alpha \leqslant 120°$ the extreme values of the interval must be increased by 1/6 and α replaced by $\alpha - 60°$, for $120° < \alpha \leqslant 180°$ they must be increased by 1/3, and α replaced by $\alpha - 120°$.

242. Note that the area of the projection of any parallelepiped is always twice the area of the projection of some triangle with vertices at the end points of three edges of the parallelepiped emanating from one of its vertices. For a rectangular parallelepiped all such triangles are congruent. The greatest area of the projection of a rectangular parallelepiped will be obtained when one of such triangles is parallel to the plane on which the parallelepiped is projected. Thus, the greatest area of the projection is equal to $\sqrt{a^2b^2+b^2c^2+c^2a^2}$.

243. Prove that the volume of such tetrahedron is less than the volume of the tetrahedron two faces of which are regular triangles with side of 1 forming a right angle,

244. (1) This statement is false. For instance, take inside the triangle ABC two points D_1 and E_1 such that the sum of the distances from D_1 to the vertices of the triangle is less than the sum of the distances from E_1 to the vertices. Now, take a point D sufficiently close to D_1 so that the sum of the distances from D to the vertices A, B, and C remains less than the sum of the distances from the point E_1. Take E inside $ABCD$ on the perpendicular to the plane ABC erected at the point E_1.

(2) This statement is true. Let us prove this. Denote by M the point of intersection of the line DE and the plane ABC. Obviously, M lies inside the triangle ABC.

The lines AM, BM, and CM separate the plane of the triangle ABC into six parts. The projection of D on the plane ABC, the point D_1, is found in one of these six parts. Depending on the position of D_1, one of the angles $\widehat{D_1MA}$, $\widehat{D_1MB}$, $\widehat{D_1MC}$ is obtuse. If the angle D_1MA is obtuse, then $\widehat{DMA}$ is also obtuse, and, hence, the angle DEA is also obtuse. Hence it follows that $|DE| < |DA|$.

245. Let $2a$ be a side of the base, h the altitude of the pyramid. Then R is equal to the radius of the circle circumscribed about the isosceles triangle with base $2a\sqrt{2}$ and altitude h, $R = \dfrac{2a^2+h^2}{2h}$; r is equal to the radius of the circle inscribed in an isosceles triangle with base $2a$ and altitude h,

$$r = \frac{a}{h}\left(\sqrt{a^2+h^2}-a\right).$$

Let

$$\frac{R}{r} = \frac{2a^2+h^2}{2a\left(\sqrt{a^2+h^2}-a\right)} = k, \quad h^2 = xa^2.$$

We will have $2 + x = 2k(\sqrt{1+x} - 1)$, whence $x^2 + 4(1 + k - k^2)x + 4 + 8k = 0$. The discriminant of this equation is equal to $16k^2(k^2 - 2k - 1)$. Thus, $k \geqslant \sqrt{2} + 1$, which was required to be proved.

246. The centres of gravity of the faces of the tetrahedron serve as the vertices of the tetrahedron similar to the given one with the ratio of similitude 1/3. Consequently, the radius of the sphere passing through the centres of gravity of the faces of the given tetrahedron is equal to $R/3$. Obviously, this radius cannot be less than the radius of the sphere inscribed in the given tetrahedron.

247. Let in the tetrahedron $ABCD$ $|AB| = b$, $|CD| = c$, the remaining edges being equal to a. If N is the midpoint of AB and M is the midpoint of CD,

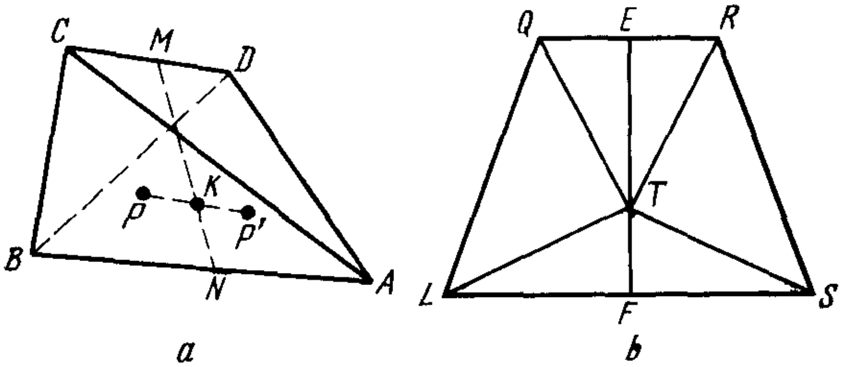


Fig. 49

then the straight line MN is the axis of symmetry of the tetrahedron $ABCD$ (Fig. 49, a). Now it is easy to prove that the point for which the sum of the distances to the vertices of the tetrahedron reaches the smallest value must lie on the line MN. Indeed, let us take an arbitrary point P and a point P' symmetric to it with respect to the line MN. Then the sums of the distances from P and P' to the vertices of the tetrahedron are equal. If K is the midpoint of PP' (K lies on MN), then in the triangles PAP', PBP', PCP', and PDP', AK, BK, CK, and DK are the respective medians, and a median of a triangle is less than the half-sum of the sides including it.

The quantity $|MN|$ is readily found:

$$|MN| = \sqrt{a^2 - \frac{b^2}{4} - \frac{c^2}{4}} = d.$$

Consider the equilateral trapezoid $LQRS$ (Fig. 49, b) in which the bases $|LS|$ and $|QR|$ are equal to b and c, respectively, and the altitude is equal to d. Let F and E be the respective midpoints of the bases LS and QR. If K is a point on MN, and T on FE, and $|FT| = |NK|$, then, obviously, the sums of the distances from K to the vertices A, B, C, and D and from T to the vertices L, S, Q, and R are equal. And in the trapezoid $LQRS$ (as well as in any convex quadrilateral) the sum of the distances to the vertices reaches the least value at the point of intersection of the diagonals and is equal to the sum of diagonals.

Answer: $\sqrt{4a^2+2bc}$.

248. Prove that the shortest way leading from the point A belonging to the circle of the greater base to the

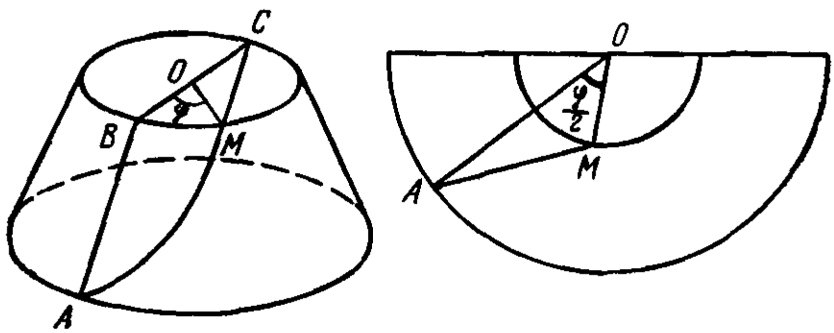


Fig. 50

diametrically opposite point C of the other base consists of the element AB and diameter BC. Its length is $2R$. Denote by r the radius of the smaller base, by O its centre. Consider the path leading from A to some point M belonging to the smaller base. The arc $\smile{AM}$ situated on the lateral surface of the cone will have the smallest length if a line segment will correspond to it on the development of the lateral surface of the cone. But this development with the angle between the generatrix and the base equal to $\pi/3$ and the radius of the base R represents a semicircle of radius $2R$. Hence, the development of a frustum of a cone is a semiannulus. Here, if to the arc $\smile{BM}$ on the base there corresponds a central angle φ, then on the development, a central angle $\frac{\varphi}{2}$ (Fig. 50) will correspond to this arc. Consequently,

$$|AM|^2 = 4R^2 + 4r^2 - 8Rr\cos\frac{\varphi}{2}, \quad |MC| = 2r\cos\frac{\varphi}{2}.$$

It remains to prove that

$$\sqrt{4R^2 + 4r^2 - 8Rr\cos\frac{\varphi}{2}} + 2r\cos\frac{\varphi}{2} \geqslant 2R.$$

This inequality is proved with the aid of obvious transformations.

249. Fix the quantities $|\mathbf{a}|$, $|\mathbf{b}|$, $|\mathbf{c}|$, denote by x, y, and z the cosines of the respective angles between $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{c}$ and $\mathbf{a}$.

Consider the difference between the left-hand and right-hand sides of the inequality in question.

We get

$$\begin{aligned}
&|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| \\
&+\sqrt{|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2|\mathbf{a}|\cdot|\mathbf{b}|x + 2|\mathbf{b}|\cdot|\mathbf{c}|y + 2|\mathbf{c}|\cdot|\mathbf{a}|z} \\
&-\sqrt{|\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}|\cdot|\mathbf{b}|x} - \sqrt{|\mathbf{b}^2| + |\mathbf{c}|^2 + 2|\mathbf{b}|\cdot|\mathbf{c}|y} \\
&-\sqrt{|\mathbf{c}|^2 + |\mathbf{a}|^2 + 2|\mathbf{c}|\cdot|\mathbf{a}|z} = f(x, y, z).
\end{aligned}$$

Note that the function $\varphi(t) = \sqrt{d+t} - \sqrt{l+t} = \dfrac{d-l}{\sqrt{d+t} + \sqrt{l+t}}$ is monotone with respect to t. This implies that $f(x, y, z)$ reaches its least value when x, y, z are equal to ± 1, that is, when the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are collinear. In this case our inequality is readily verified.

250. Let the straight line MN intersect D_1C_1 at the point L. Set: $|AM| = x$, $|BN| = y$. It follows from the hypothesis that $x > a$, $y > a$. Projecting all the points on the plane ABB_1A_1, we find $\dfrac{|C_1L|}{|LD_1|} = \dfrac{a}{x-a}$, and projecting them on the plane $ABCD$, we find $\dfrac{|C_1L|}{|LD_1|} =$

$\frac{y-a}{a}$. Consequently, $\frac{a}{x-a}=\frac{y-a}{a}$, whence $xy = (x+y)a$. But $(x+y)^2 \geqslant 4xy$. Hence, $xy \geqslant 4a^2$.

Now, we get $|MN|^2 = x^2 + y^2 + a^2 = (x+y)^2 - 2xy + a^2 = \frac{(xy)^2}{a^2} - 2xy + a^2 = \frac{1}{a^2}(xy - a^2)^2 \geqslant 9a^2$. The least value of $|MN|$ is equal to $3a$.

251. If x is the length of two other sides of the rectangle, then the volume of the pyramid is equal to $\frac{ax}{3}\sqrt{b^2-\frac{a^2}{4}-\frac{x^2}{4}}$. The greatest value of the volume will be for $x=\sqrt{\frac{4b^2-a^2}{2}}$, it equals $\frac{a(4b^2-a^2)}{12}$.

252. Let M be a point on the line AB_1, N on the line BC_1, M_1 and N_1 the respective projections of M and N on the plane $ABCD$. Setting $|BM_1| = x$, $|BN_1| = y$, we get

$$|M_1N_1| = \sqrt{x^2+y^2}, \quad |MN| = \sqrt{x^2+y^2+(a-x-y)^2}.$$

By the hypothesis, $|MN| = 2|M_1N_1|$, consequently, $(a-x-y)^2 = 3(x^2+y^2)$. Let $x^2+y^2 = u^2$, $x+y = v$, then $2u^2 - v^2 \geqslant 0$, and since $u^2=\frac{1}{3}(a-v)^2$, replacing u^2 in the inequality relating u and v, we obtain the following inequality for v: $v^2 + 4av - 2a^2 \leqslant 0$ whence $a(2+\sqrt{6}) \leqslant v \leqslant a(\sqrt{6}-2)$. We now find the least value of $|MN|$, it is equal to $2a(\sqrt{3}-\sqrt{2})$.

253. Consider the cube $ABCDA_1B_1C_1D_1$ with an edge $2R$. Arrange the axes of the given cylinders on the lines AA_1, DC, B_1C_1.

(a) The centre of the cube is at a distance of $R\sqrt{2}$ from all the edges of the cube. Any point in space is located at a distance greater than $R\sqrt{2}$ from at least one of the edges AA_1, DC, B_1C_1. This follows from the fact that the cylinders with axes AA_1, DC, B_1C_1 and radii $R\sqrt{2}$ have the only common point, the centre of the cube. Consequently, the radius of the smallest ball touching all the three cylinders is equal to $R(\sqrt{2}-1)$.

(b) If K, L, and M are the respective midpoints of the edges AA_1, DC, and B_1C_1, then the straight line passing through the centre of the cube perpendicular to the plane KLM is found at a distance of $R\sqrt{2}$ from the lines AA_1, DC, and B_1B; KLM is a regular triangle, its centre coincides with the centre of the cube. Hence it follows that any straight line intersecting the plane KLM is situated from at least one vertex of the triangle KLM at a distance not exceeding the radius of the circle circumscribed about it which is equal to $R\sqrt{2}$. Thus, the radius of the greatest cylinder touching the three given cylinders and satisfying the conditions of the problem is equal to $R(\sqrt{2}-1)$.

254. Let $ABCD$ be the tetrahedron of the greatest volume, O the centre of the given spheres. Each line segment joining O to the vertex of the tetrahedron must be perpendicular to the face opposite to this vertex. If, for instance, AO is not perpendicular to the plane BCD, then on the surface of the sphere on which the point A lies it is possible to find points lying at greater distances than the point A does. (This reasoning remains, obviously, true if A, B, C, and D lie on the surfaces of different spheres and even not necessarily concentric ones.) Hence it follows that the opposite edges of the tetrahedron $ABCD$ are pairwise perpendicular. Let, further, the points A and B lie on the sphere of radius $R=\sqrt{10}$, and C and D on the sphere of radius $r=2$. Denote by x and y the respective distances from O to AB and CD.

Through AB, draw a section perpendicular to CD. Denote by K the point of intersection of this plane and CD. Taking into consideration the properties of our tetrahedron $ABCD$, it is easy to prove that $|AK| = |BK|$, O is the point of intersection of the altitudes of the triangle ABK. Draw the altitudes KL and AM (Fig. 51). From the similarity of the triangles ALO and OKM we find $|OM| = \frac{xy}{R}$. Further, $|AB| = 2\sqrt{R^2-x^2}$, and from the similarity of the triangles AOL and AMB we get

$$\frac{R}{\sqrt{R^2-x^2}}=\frac{2\sqrt{R^2-x^2}}{R+\frac{xy}{R}},$$

whence $2x^2 + xy = R^2$. Proceeding in the same way, we get the equation $2y^2 + xy = r^2$. From the system of equations

$$\begin{cases} 2x^2+xy=10, \\ 2y^2+xy=4 \end{cases}$$

we find $x = 2$, $y = 1$. The volume of the tetrahedron $ABCD$ will be equal to $6\sqrt{2}$.

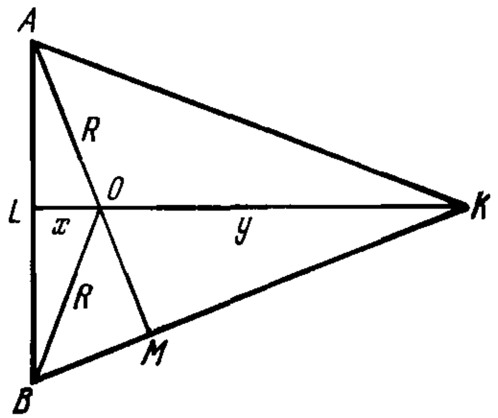


Fig. 51

255. Let A denote the vertex of the trihedral angle whose plane angles are right angles, B the vertex of the other angle. On the line segment AB take a point M such that $2\,|AM| = |MB|$. Through the point M pass a plane perpendicular to AB. This plane will cut each of the two trihedral angles in a regular triangle with side $b = a\sqrt{\frac{2}{3}}$. In Fig. 52, *a*, the triangle PQR corresponds to the section of the trihedral angle with the vertex A. The face BCD cuts off the pyramid $QFKL$ from the pyramid $APQR$ (the position of the point F is clear from Fig. 52, *b*). The volume of this pyramid is proportional to the product $|QK|\cdot|QL|\cdot|QF|$. The quantity $|QF|$, obviously, reaches the greatest value for $\alpha = \pi/3$, where $\alpha = \widehat{CMQ}$. Let us prove that $|KQ|\cdot|QL|$ reaches

the greatest value also for $\alpha = \pi/3$. Since KL is tangent to the circle inscribed in PQR, the perimeter of the triangle KQL is constant and is equal to b. We set $|KQ| =$

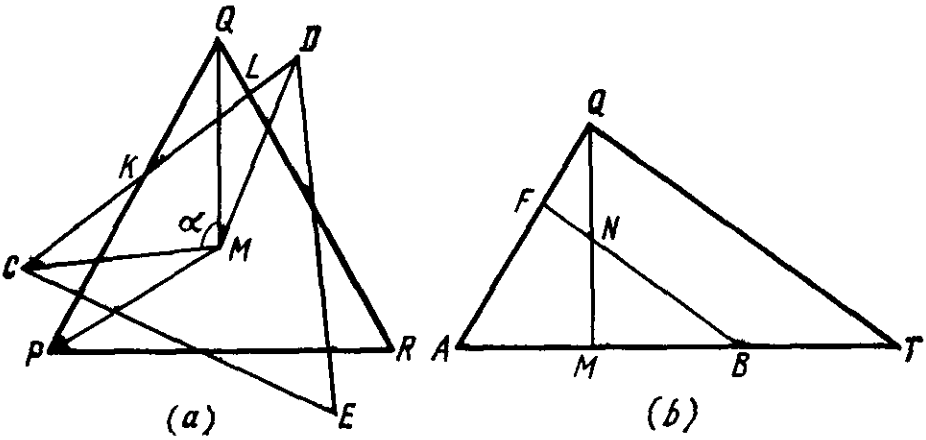


Fig. 52

x, $|QL| = y$, then $KL = b - x - y$. Write the theorem of cosines for the triangle KQL:

$$(b - x - y)^2 = x^2 + y^2 - xy \Rightarrow b^2 - 2b(x + y) + 3xy$$
$$= 0 \Rightarrow b^2 - 4b\sqrt{xy} + 3xy \geqslant 0.$$

Consequently, either $\sqrt{xy} \leqslant \dfrac{b}{3}$, or $\sqrt{xy} \geqslant b$. But $0 \leqslant x \leqslant \dfrac{b}{2}$ and $0 \leqslant y \leqslant \dfrac{b}{2}$. Hence, $\sqrt{xy} \leqslant \dfrac{b}{3}$. Equality is obtained if $x = y = \dfrac{b}{3}$.

Thus, the volume of the pyramid $QKLF$ is the greatest for $\alpha = \pi/3$. Here, $|KQ| = |QL| = \dfrac{b}{3} = \dfrac{a}{3}\sqrt{\dfrac{2}{3}}$. Further, for $\alpha = \pi/3$, N is the midpoint of QM (Fig. 52, b). Drawing QT parallel to FB, we get $|BT| = |MB|$. Thus,

$$\frac{|AF|}{|FQ|} = \frac{|AB|}{|BT|} = \frac{3}{2}, \quad |QF| = \frac{2}{5}|AQ|.$$

The volume of the pyramid $APQR$ is found readily, it is equal to $\frac{a^3\sqrt{3}}{54}$. Three pyramids equal to the pyramid $QFKL$ are cut off the pyramid $APQR$.

The volume of each of them amounts to $\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{2}{5}=\frac{2}{45}$ the volume of the pyramid $APQR$. Thus, for $\alpha=\pi/3$ we get the "remainder" of the pyramid $APQR$, that is, a polyhedron having the volume

$$\frac{a^3\sqrt{3}}{54}\left(1-\frac{2}{15}\right)=\frac{13a^3\sqrt{3}}{810}.$$

Reasoning exactly in the same manner, we get that for $\alpha=\pi/3$ from the pyramid $BCDE$ there will remain a polyhedron of the smallest volume, and the volume of this polyhedron will be $\frac{11a^3\sqrt{3}}{324}$.

Adding the obtained volumes, we get the *answer*: $\frac{a^3\sqrt{3}}{20}$.

256. Setting $|BD|=2x$, it is easy to find

$$V=V_{ABCD}=\frac{x\,|1-2x^2|\sqrt{3-4x^2}}{6(1-x^2)}.$$

Making the substitution $u=1-x^2$, and then $w=4u+1/u$, we get

$$\begin{aligned}(6V)^2&=\frac{x^2(1-2x^2)^2(3-4x^2)}{(1-x^2)^2}\\&=\frac{(1-u)(2u-1)^2(4u-1)}{u^2}\\&=\left(5-\frac{1}{u}-4u\right)\left(4u+\frac{1}{u}-4\right)\\&=(5-w)(w-4)=-w^2+9w-20.\end{aligned}$$

The greatest value is attained for $w = 9/2$, whence

$$x=\sqrt{1-u}=\sqrt{1-\frac{9\pm\sqrt{17}}{16}}.$$

Answer: the greatest value of V_{ABCD} equals $\frac{1}{12}$.

257. Let x denote the radius of the ball, $V(x)$ the sum of the volume of the part of the ball situated outside the tetrahedron and the part of the tetrahedron outside the ball. It is easy to see that $V'(x) = S_1(x) - S_2(x)$, where $S_1(x)$ is the surface area of the part of the ball outside the tetrahedron, $S_2(x)$ is the surface area of the part of the ball enclosed inside the tetrahedron. Minimum is reached for $S_1(x) = S_2(x)$, whence $x = a\frac{1}{3}\sqrt{\frac{2}{3}}$.

258. Let a, b, c be the sides of the base, $p=\frac{a+b+c}{2}$, r the radius of the inscribed circle, x, y, z the distances from the foot of the altitude of the pyramid to the sides a, b, c, and h the altitude of the pyramid. Then

$$S_{\text{lat}}=\frac{1}{2}a\sqrt{h^2+x^2}+\frac{1}{2}b\sqrt{h^2+y^2}+\frac{1}{2}c\sqrt{h^2+z^2}.$$

Note that the function $f(x) = \sqrt{h^2+x^2}$ is concave (convex downward). And for such functions the following inequality is valid:

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) + \ldots + \alpha_n f(x_n)$$
$$\geqslant f(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n),$$
$$\alpha_i \geqslant 0,\quad i = 1, 2, \ldots, n,\quad \alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$$

Let us take advantage of this inequality. We get

$$S_{\text{lat}}=p\left(\frac{a}{2p}\sqrt{h^2+x^2}+\frac{b}{2p}\sqrt{h^2+y^2}+\frac{c}{2p}\sqrt{h^2+z^2}\right)$$
$$\geqslant p\sqrt{h^2+\left(\frac{a}{2p}x+\frac{b}{2p}y+\frac{c}{2p}z\right)^2}$$
$$=p\sqrt{h^2+\frac{S_{\text{base}}^2}{4p^2}}=p\sqrt{h^2+r^2},$$

which was required to be proved.

259. If O is the centre of the circle, L is the projection of N on the plane of the base, then the point M must lie on the line segment LO since M is a point of the circle nearest to N. On the other hand, since N is a point of the diagonal of the face nearest to M, MN is perpendicular to this diagonal, and, hence, KN is also perpendicular to this diagonal, where K is the projection of M on the face containing this diagonal (Fig. 53).

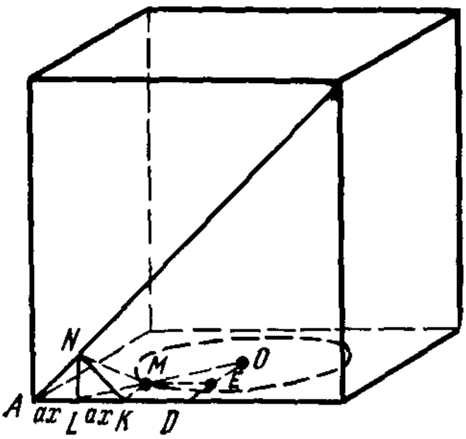


Fig. 53

Let $|AL| = ax$, ANK is an isosceles right triangle, consequently, $|LK| = |AL| = ax$,

$$|MK| = |OD| \frac{|LK|}{|LD|} = \frac{ax}{1-2x},$$

$$|KD| = \frac{a}{2}(1-4x).$$

Writing the Pythagorean theorem for $\triangle MOE$ (ME is parallel to AD), we get the following equations for x:

$$\frac{(1-4x)^2}{4} + \left(\frac{1}{2} - \frac{x}{1-2x}\right)^2 = \frac{25}{144}$$

$$\Leftrightarrow [6(1-4x)(1-2x)]^2 + [6(1-4x)]^2 + [5(1-2x)]^2.$$

Making the substitution $5^2 = 3^2 + 4^2$ in the right-hand side and transposing it to the left, we get

$$[6(1-4x)(1-2x)]^2$$
$$-[3(1-2x)]^2+[6(1-4x)]^2-[4(1-2x)]^2=0$$
$$\Leftrightarrow 9(1-2x)^2(1-8x)(3-8x)+4(5-16x)(1-8x)$$
$$=0 \Leftrightarrow (1-8x)[9(1-2x)^2(3-8x)$$
$$+4(5-16x)]=0.$$

It is easy to see that the point K must lie to the left of the point D, that is, $0 < x < 1/4$, hence, the expression in the square brackets is not equal to zero, $x = 1/8$.

Answer: $a\dfrac{\sqrt{34}}{24}$.

260. (a) Let $|SC| = d$; a, b, and c the sides of the triangle ABC, h_a, h_b, h_c the altitudes of the triangle ABC, and s its area. Then

$$\sin\alpha=\frac{h_a}{\sqrt{a^2+b^2}}, \quad \sin\beta=\frac{h_b}{\sqrt{d^2+a^2}}, \quad \sin\gamma=\frac{h_c}{\sqrt{d^2+h_c^2}}.$$

Thus, we get for d the equation

$$\frac{\sqrt{d^2+b^2}}{h_a}+\frac{\sqrt{d^2+a^2}}{h_b}=1+\frac{\sqrt{d^2+h_c^2}}{h_c}.$$

Multiplying this equation by $2s$, we get

$$a\sqrt{d^2+b^2}+b\sqrt{d^2+a^2}=2s+\sqrt{c^2d^2+4s^2}. \qquad (1)$$

Multiplying and dividing both sides of (1) by the differences of the corresponding quantities (assuming that $\hat{A} \neq \hat{B}$), we get

$$\frac{a^2-b^2}{a\sqrt{d^2+b^2}-b\sqrt{d^2+a^2}}=\frac{c^2}{\sqrt{c^2d^2+4s^2}-2s},$$

whence

$$ac^2\sqrt{d^2+b^2}-bc^2\sqrt{d^2+a^2}=(a^2-b^2)\left(\sqrt{c^2d^2+4s^2}-2s\right) \qquad (2)$$

Multiplying (1) by $b^2 - a^2$ and adding the result to (2), we obtain

$$a(b^2+c^2-a^2)\sqrt{a^2+b^2}+b(b^2-a^2-c^2)\sqrt{d^2+a^2}$$
$$=4s(b^2-a^2).$$

With the aid of the theorems of cosines and sines, the last equation is transformed as follows

$$\cos A\cdot\sqrt{d^2+b^2}-\cos B\cdot\sqrt{d^2+a^2}=\frac{b^2-a^2}{2H}. \qquad (3)$$

Transform the right-hand member of Equation (3) as follows:

$$\frac{b^2-a^2}{2R}=2R(\sin^2 B-\sin^2 A)=2R\sin(A+B)\sin(B-A),$$

now, multiplying both sides of (3) by $\cos A\cdot\sqrt{d^2+b^2}+\cos B\cdot\sqrt{d^2+a^2}$, we get the equation

$$(\cos^2 A-\cos^2 B)\,d^2+b^2\cos^2 A-a^2\cos^2 B$$
$$=2R\sin(A+B)\sin(B-A)$$
$$\times(\cos A\cdot\sqrt{d^2+b^2}+\cos B\cdot\sqrt{d^2+a^2}). \qquad (4)$$

In Equation (4) we see $\cos^2 A-\cos^2 B=\sin(A+B)\sin(B-A)$, $b^2\cos^2 A-a^2\cos^2 B=4R^2\sin(B+A)\sin(B-A)$. Consequently, after reduction, Equation (4) is transformed to

$$\cos A\cdot\sqrt{d^2+b^2}+\cos B\cdot\sqrt{d^2+a^2}=\frac{d^2}{2R}+2R. \qquad (4')$$

Adding (3) and (4′), we get

$$2\cos A\cdot\sqrt{d^2+b^2}=\frac{d^2}{2R}+2R(\sin^2 B+\cos^2 A),$$

whence

$$(\sqrt{d^2+b^2}-2R\cos A)^2=0,$$

$$d^2 = 4R^2(\cos^2 A - \sin^2 B)$$

$$= 4R^2 \cos(A+B)\cos(A-B).$$

Thus,

$$|SC| = 2R\sqrt{\cos(A+B)\cos(A-B)}.$$

The problem has a solution if $A+B<90°$, that is, in the triangle ABC the angle C is obtuse.

(b) Let us take advantage of the notation used in Item (a). Then our inequality is rewritten in the form

$$\frac{\sqrt{d^2+a^2}}{h_b}+\frac{\sqrt{d^2+b^2}}{h_a}-\frac{\sqrt{d^2+h_c^2}}{h_c}\geqslant 1.$$

If the angle C is acute, then the right-hand side, as it follows from Item (a), is never equal to 1, consequently, the inequality takes place, since it is fulfilled for $d=0$. And if C is an obtuse angle (or it is equal to 90°), then the right-hand side is equal to 1 for the unique value of d (if C is a right angle, then $d=0$). But for $d=0$ and sufficiently large values of d the inequality is obvious (for large d's it follows from the triangle inequality), consequently, if for some value of d the left-hand side were less than unity, then the left-hand side would take on the value equal to unity for two different values of d.

261. Let $ABCD$ be the given tetrahedron. On the edges BC and BD take points M and N and solve the following problem: for what position of the points M and N does the radius of the smallest circle enclosing the triangle AMN (we consider the circles lying in the plane AMN) reach the least value? (Obviously, the radius of the smallest hole cannot be less than this radius. For this purpose, it suffices to consider the instant of passing of the tetrahedron through the hole when two vertices of the tetrahedron are found on one side of the plane of the hole, the third vertex on the other side, and the fourth in the plane of the hole.)

Suppose that the points M and N correspond to the desired triangle. Suppose that this triangle is acute.

Then the smallest circle containing this triangle coincides with the circumscribed circle. Circumscribe a circle about the triangle AMN and consider the solid obtained by revolving the arc AMN of this circle about the chord AN. The straight line BC must be tangent to the surface of this solid. Otherwise, on BC we could take a point M_1 such that the radius of the circle circumscribed about the triangle AM_1N would be less than the radius of the circle circumscribed about the triangle AMN. The more so, BC must be tangent to the surface of the sphere passi g through A, M, and N having the centre in the plane AMN. The straight line BD must also touch this sphere exactly in the same manner. Consequently, $|BM| = |BN|$. Set $|BM| = |BN| = x$. Let K denote the midpoint of MN, L the projection of B on the plane AMN (L lies on the extension of AK). The foregoing implies that LM and LN are tangents to the circle circumscribed about the triangle AMN. This triangle is isosceles, $|AM| = |AN| = \sqrt{x^2 - x + 1}$, $|MN| = x$. If $\widehat{MAN} = \alpha$, then

$$\cos\alpha = \frac{x^2 - 2x + 2}{2(x^2 - x + 1)}, \quad \sin\alpha = \frac{x\sqrt{3x^2 - 4x + 4}}{2(x^2 - x + 1)},$$

$$|LK| = |MK| \tan\alpha = \frac{x^2\sqrt{3x^2 - 4x + 4}}{2(x^2 - 2x + 2)}.$$

Consider the triangle AKB, $\widehat{AKB} = \beta > 180°$; $\cos\beta = \dfrac{3x - 2}{\sqrt{3(3x^2 - 4x + 4)}}$, $|LK| = -|KB|\cos\beta = \dfrac{x(2 - 3x)}{2\sqrt{3x^2 - 4x + 4}}$. Equating two expressions for $|LK|$, we get for x, after simplifications, the equation

$$3x^3 - 6x^2 + 7x - 2 = 0. \tag{1}$$

The radius of the circle circumscribed about the triangle AMN, will be

$$R = \frac{x^2 - x + 1}{\sqrt{3x^2 - 4x + 4}}.$$

(It is possible to show that if AMN is a right triangle, then its hypotenuse is not less than $\sqrt{15 - 10\sqrt{2}} > 0.9$.) Let us show that our tetrahedron can go through the hole of the found radius.

On the edges CB and CA mark points L and P such that $|CL| = |CP| = |BM| = |BN| = x$, where x satisfies the equation (1).

Place the tetrahedron on the plane containing the given hole so that M and N are found on the boundary of the hole. We will rotate the tetrahedron about the line MN until the edge AB, passing the hole, becomes parallel to our plane. Then, retaining AB parallel to this plane, we displace the tetrahedron $ABCD$ so that the points P and L get on the boundary of the hole. And, finally, we shall rotate the tetrahedron about PL until the edge DC goes out from the hole. (The tetrahedron will turn out to be situated on the other side of our plane, the face ABC lying in this plane.)

Answer: the radius of the smallest hole $R = \dfrac{x^2-x+1}{\sqrt{3x^2-4x+4}}$, where x is the root of the equation $3x^3 - 6x^2 + 7x - 2 = 0$. The relevant computations yield the following approximate values: $x \approx 0.3913$, $R \approx 0.4478$ with an error not exceeding 0.00005.

Section 4

262. Let S denote the vertex of the angle. Take points A, B, and C on the edges such that $|SA| = |SB| = |SC|$. The bisectors of the angles ASB and BSC pass through the midpoints of the line segments AB and BC, while the bisector of the angle adjacent to the angle CSA is parallel to CA.

264. $\left[\dfrac{1}{2\sin\frac{\alpha}{2}}\right]$, if $\dfrac{1}{2\sin\frac{\alpha}{2}}$ is not a whole number, $\dfrac{1}{2\sin\frac{\alpha}{2}} - 1$, if $\dfrac{1}{2\sin\frac{\alpha}{2}}$ is a whole number, where $[x]$ is an integral part of x.

265. We shall regard the given lines as the coordinate axes. Let the straight line make angles α, β, and γ with these axes. Then the projections of the vectors $\overrightarrow{OA_1}$, $\overrightarrow{OB_1}$, and $\overrightarrow{OC_1}$ on the axes OA, OB, and OC will be respectively equal to $a \cos 2\alpha$, $a \cos 2\beta$, and $a \cos 2\gamma$, $a = |OA|$. Consequently, the point M of intersection of the planes passing through A_1, B_1, and C_1 respectively perpendicular to OA, OB, and OC will have the coordinates ($a \cos 2\alpha$, $a \cos 2\beta$, and $a \cos 2\gamma$). The set of points with the coordinates ($\cos^2 \alpha$, $\cos^2 \beta$, and $\cos^2 \gamma$) is a triangle with vertices at the end points of the unit vectors of the axes. Consequently, the sought-for locus of points is also a triangle whose vertices have the coordinates ($-a$, $-a$, a); ($-a$, a, $-a$); (a, $-a$, $-a$).

266. Denote the given lines by l_1 and l_2. Through l_1 pass a plane p_1 parallel to l_2, and through l_2 a plane p_2 parallel to l_1. It is obvious that the midpoints of the line segments with the end points on l_1 and l_2 belong to the plane p parallel to p_1 and p_2 and equidistant from p_1 and p_2. (It is possible to show that if we consider all kinds of such line segments, then their midpoints will entirely fill up the plane p.) Project now these line segments on the plane p parallel to the given plane. Now, their end points will lie on two straight lines which are the projections of the lines l_1 and l_2, and the projections themselves will turn out to be parallel to the given line of the plane p representing the line of intersection of the plane p and the given plane. Hence it follows that the required locus of points is a straight line.

267. (a) The whole space.

(b) Proceeding exactly in the same way as in Problem 266, we can prove that the locus of points dividing in a given ratio all possible line segments parallel to the given plane with the end points on the given skew lines is a straight line. Applying this statement twice (first, find the locus of midpoints of sides AB, and then the locus of centres of gravity of triangles ABC), prove that in this case the locus of centres of gravity of triangles ABC is a straight line.

268. Through the common perpendicular to the straight lines, pass a plane p perpendicular to l_3. Let the line NM intersect l_3 at point L; N_1, M_1, L_1 be the respective points of intersection of the lines l_1, l_2, l_3 with the common perpendicular, N_2, M_2 the projections of N and M on the passed plane, α and β the angles made by the lines l_1 and l_2 with this plane, K the midpoint of

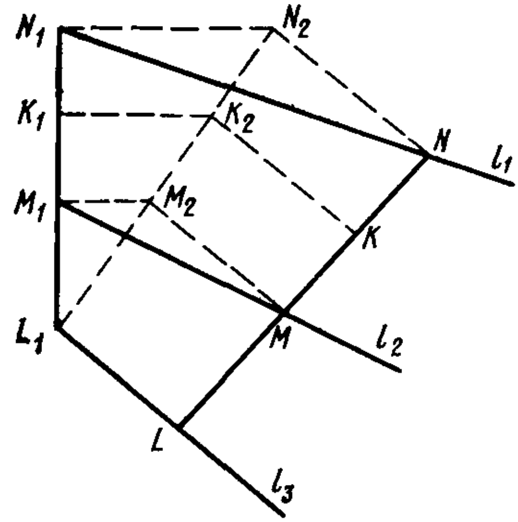


Fig. 54

NM, K_1 and K_2 the projections of K on the common perpendicular and on the plane p (Fig. 54). We have

$$\frac{|KK_2|}{|K_1K_2|}=\frac{|NN_2|+|MM_2|}{|N_2N_1|+|M_2M_1|}$$
$$=\frac{|N_2N_1|\tan\alpha+|M_2M_1|\tan\beta}{|N_1N_1|+|M_2M_1|}$$
$$=\frac{|N_1L_1|\tan\alpha+|M_1L_1|\tan\beta}{|M_1L_1|+|M_1L_1|}=\text{const},$$

hence, the point K describes a straight line.

269. Let us introduce a rectangular coordinate system, choosing the origin at the point A. Let $\mathbf{e}_1$ (a_1, b_1, c_1), $\mathbf{e}_2$ (a_2, b_2, c_2), . . ., $\mathbf{e}_n$ (a_n, b_n, c_n) be unit vectors parallel to the given lines, $\mathbf{e}$ (x, y, z) a unit vector parallel to the line satisfying the conditions of the problem. Thus, we get for $\mathbf{e}$ the following equation

$$|a_1x+b_1y+c_1z|+|a_2x+b_2y+c_2z|+\ldots$$
$$+|a_nx+b_ny+c_nz|=\text{const}.$$

It is now easily seen that the locus of termini of the vector e will be the set of circles or parts thereof situated on the surface of the unit sphere with centre at A.

270. Place equal loads at the points A, B, C, A_1, B_1, and C_1. Then the centre of gravity of the obtained system of loads will coincide with the centre of gravity of the triangle with vertices at the midpoints of the line segments AA_1, BB_1, CC_1.

On the other hand, the centre of gravity of this system coincides with the midpoint of the line segment GH, where G is the centre of gravity of the triangle ABC, H the centre of gravity of the three loads found at A_1, B_1, and C_1.

With a change in A_1, B_1, and C_1 the point H moves in the straight line l, and the point G remains fixed. Hence, the point M, which is the midpoint of GH, will describe a straight line parallel to l.

271. Through A draw a straight line t parallel to l. The sought-for locus of points represents a cylindrical surface, except for l and t, in which l and t are diametrically opposite elements.

272. Let us first prove that if the line MK is tangent to the sphere β, then it is also tangent to the sphere α.

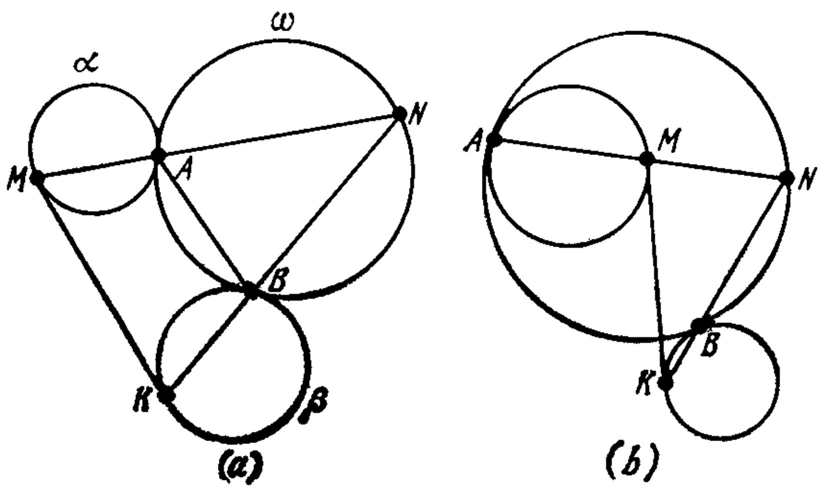


Fig. 55

Consider the section of the given spheres by the plane passing through points M, K, A, B, and N (Fig. 55). The angle MKB is measured by half the arc KB enclosed inside this angle, consequently, $\widehat{MKB} = \widehat{BAN}$, since the angle measures of the arcs KB and BN are equal (we take the arcs situated on different sides of the line KN if the tangency is external (Fig. 55, *a*) and situated on one side if the tangency is internal (Fig. 55, *b*)). Hence it follows that $\widehat{AMK} = \widehat{ABN}$ or $\widehat{AMK} = 180° - \widehat{ABN}$, and this means that $\widehat{AMK}$ is measured by half $\smile{AM}$, since the corresponding arcs AM and AN have the same angle measure, that is, MK touches the circle along which the considered section cuts the sphere α.

It is now possible to prove that the locus of points M is a circle.

273. Let A and B denote the given points, C the point of intersection of the line AB with the given plane, M the point of tangency of a ball with the plane. Since $|CM|^2 = |CA| \cdot |CB|$, M lies on the circle with centre at the point C and radius $\sqrt{|CA| \cdot |CB|}$. Consequently, the centre of the sphere belongs to the lateral surface of the right cylinder whose base is this circle. On the other hand, the centre of the sphere belongs to the plane passing through the midpoint of AB perpendicular to AB. Thus, the sought-for locus of points is the line of intersection of the lateral surface of a cylinder and a plane (this line is called the *ellipse*).

274. Denote by O_1, O_2 and R_1, R_2 the centres and radii of the given spheres, respectively; M is the midpoint of a common tangent. Then, it is easy to see that

$$|O_1M|^2 - |O_2M|^2 = R_1^2 - R_2^2,$$

and, consequently, M lies in the plane perpendicular to the line segment O_1O_2 and cutting this segment at a point N such that

$$|O_1N|^2 - |O_2N|^2 = R_1^2 - R_2^2.$$

Let us see what is the range of variation of the quantity $|NM|$. Let $|O_1O_2| = a$ and $R_1 \geqslant R_2$, then

$$|O_1N| = \frac{1}{2}\left(\frac{R_1^2 - R_2^2}{a} + a\right).$$

If $2x$ is the length of the common tangent, whose midpoint is M, then

$$|MN|^2 = |O_1M|^2 - |O_1N|^2 = x^2 + R_1^2 - \frac{1}{4}\left(\frac{R_1^2 - R_2^2}{a} + a\right).$$

Now, if $a \geqslant R_1 + R_2$, then the quantity $4x^2$ changes within the interval from $a^2 - (R_1 + R_2)^2$ to $a^2 - (R_1 - R_2)^2$, and, hence, in this case the locus of points M will be an annulus whose plane is perpendicular to O_1O_2, and the centre is found at the point N, the inner radius is equal to

$$\frac{1}{2}(R_1 - R_2)\sqrt{1 - \frac{(R_1 + R_2)^2}{a^2}},$$

and the outer to

$$\frac{1}{2}(R_1 + R_2)\sqrt{1 - \frac{(R_1 - R_2)^2}{a^2}}.$$

And if $a < R_1 + R_2$, that is, the spheres intersect, then the inner radius of the annulus will be equal to the radius of the circle of their intersection, that is, it will be

$$\frac{1}{2a}\sqrt{(a + R_1 + R_2)(a + R_1 - R_2)(a + R_2 - R_1)(R_1 + R_2 - a)}.$$

275. Denote by A and B the points of tangency of the lines l_1 and l_2 with the sphere, and by K the point of tangency of the line MN with the sphere. We will have

$$|AM| = |MK|, \quad |BN| = |NK|.$$

Project l_1 and l_2 on the plane perpendicular to AB. Let A_1, M_1, N_1, and K_1 denote the respective projections of the points A (and also B), M, N, and K. Obviously,

$$\frac{|A_1M_1|}{|AM|} = p, \quad \frac{|A_1N_1|}{|BN|} = q,$$

where p and q are constants. Let now d and h be the distances from K_1 to the straight lines A_1M_1 and A_1N_1.

We have

$$\frac{d}{h}=\frac{\frac{1}{2}\mid A_1M_1\mid d}{\frac{1}{2}\mid A_1N_1\mid h}\cdot\frac{\mid A_1N_1\mid}{\mid A_1M_1\mid}=\frac{S_{A_1M_1K_1}}{S_{A_1N_1K_1}}\cdot\frac{\mid A_1N_1\mid}{\mid A_1M_1\mid}$$

$$=\frac{\mid M_1K_1\mid}{\mid N_1K_1\mid}\cdot\frac{\mid A_1N_1\mid}{\mid A_1M_1\mid}=\frac{\mid MK\mid}{\mid NK\mid}\cdot\frac{\mid A_1N_1\mid}{\mid A_1M_1\mid}$$

$$=\frac{\mid AM\mid}{\mid A_1M_1\mid}\cdot\frac{\mid A_1N_1\mid}{\mid BN\mid}=\frac{q}{p}.$$

Thus, the ratio of the distances from the point K_1 to two given straight lines in the plane is constant. This means that the point K_1 belongs to one of the two straight lines passing through the point A_1. And the sought-for locus of points represents two circles on the surface of the given sphere. These circles are obtained when the sphere is cut by two planes passing through the lines described by the point K_1 and the straight line AB. The points A and B themselves are excluded.

279. Let BK denote the altitude of the triangle ABC, H the point of intersection of the altitudes of the triangle ABC, BM the altitude of the triangle DBC, N the point of intersection of the altitudes in the triangle DBC. Prove that N is the projection of the point H on the plane DBC.

Indeed, KM is perpendicular to DC, since BM is perpendicular to DC, and KM is the projection of BM on the plane ADC. Thus, the plane KMB is perpendicular to the edge DC, consequently, HN is perpendicular to DC. Exactly in the same way, HN is perpendicular to the edge DB. Hence, HN is perpendicular to the plane DBC. It is not difficult to prove now, that N lies in the plane passing through AD perpendicular to BC.

The required locus of points represents a circle with diameter HL, where L is the foot of the altitude dropped from A on BC whose plane is perpendicular to the plane ABC.

283. Denote by P and Q the points of intersection of the opposite sides of the quadrilateral $ABCD$. If the section by the plane of the lateral surface of the pyramid $ABCDM$ is a parallelogram, then the plane of the

section must be parallel to the plane PQM, the sides of the parallelogram being parallel to the straight lines PM and QM. Hence, in order for a section to be a rectangle, the angle PMQ must be equal to 90°, that is, M lies on the surface of the sphere with diameter PQ. (Thus, Item (a) has been solved.)

(b) Denote by K and L the points of intersection of the diagonals of the quadrilateral $ABCD$ and the straight line PQ. Since the diagonals of the parallelogram obtained by cutting the lateral surface of the pyramid $ABCDM$ by a plane will be parallel to the lines MK and ML, this parallelogram will be a rhombus if $\widehat{KML} = 90°$, that is, M lies on the surface of a sphere with diameter KL.

(c) Items (a) and (b) imply that the locus of points M will be a circle which is the intersection of two spheres of diameters PQ and KL.

(d) The locus of points is a conical surface with vertex at the point of intersection of the diagonals of the quadrilateral $ABCD$ whose directing curve is a circle from the preceding item.

284. If K and L are the midpoints of BC and AM, O the centre of the sphere circumscribed about $ABCM$, then, since G is the midpoint of LK and OG is perpendicular to LK, $|OL| = |OK|$. Hence it follows that $|AM| = |BC|$, that is, M lies on the surface of the sphere of radius BC centred at A.

Let, further, N be the centre of gravity of the triangle ABC, O_1 the centre of the circle circumscribed about the triangle ABC, G_1 the projection of G on the plane ABC. Since, by the hypothesis, OG is perpendicular to AK, O_1G_1 is also perpendicular to AK. Hence, G lies in the plane passing through O_1 and perpendicular to AK. Hence, since

$$|NG| = \frac{1}{4} |NM|,$$

it follows that the point M also lies in the plane perpendicular to AK.

Thus, the sought-for locus of points represents the line of intersection of a sphere and a plane, that is, generally speaking, is a circle.

285. Introduce a rectangular Cartesian coordinate system taking for O the vertex of the trihedral angle and

directing the axes along the edges of this angle. Let the plane of the circle make angles α, β, and γ with the coordinate planes XOY, YOZ, and ZOX, respectively. Then the point O_1 (the centre of the circle) will have the coordinates ($R \sin \beta$, $R \sin \gamma$, $R \sin \alpha$), where R is the radius of the circle. From the origin draw a straight line perpendicular to the plane of the circle. This line will make angles β, γ, and α with the coordinate axes. Consequently,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

and, hence,

$$|OO_1|^2 = R^2 (\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma) = 2R^2.$$

Thus, the point O_1 lies on the surface of the sphere with centre at O and radius $R\sqrt{2}$. On the other hand, the distance from O_1 to the coordinate planes does not exceed R.

Consequently, the sought-for set represents a spherical triangle bounded by the planes $x = R$, $y = R$, $z = R$ on the surface of the sphere $|OO_1| = R\sqrt{2}$, situated in the first octant.

286. Let the spider be found in the vertex A of the cube $ABCDA_1B_1C_1D_1$. Consider the triangle DCC_1. It is rather easy to prove that the shortest path joining A to any point inside the triangle DCC_1 intersects the edge DC. In this case, if the faces $ABCD$ and DCC_1D_1 are "developed" so as to get a rectangle made from two squares $ABCD$ and DCC_1D_1, then the shortest path will represent a segment of a straight line. Consequently, the arc of a circle with radius of 2 cm whose centre is found at the point A of the development situated inside the triangle DCC_1 will be part of the boundary of the sought-for locus of points. The entire boundary consists of six such arcs and separates the surface of the cube into two parts. The part which contains the vertex A together with the boundary is just the required locus of points.

287. We take the edges of the trihedral angle for the coordinate axes. Let (x, y, z) be the coordinates of the vector $\overrightarrow{OA}$, (x_i, y_i, z_i) the coordinates of the ith section of the polygonal line. Each section of the polygonal line is regarded as a vector. Then

$$x = \sum x_i, \quad y = \sum y_i, \quad z = \sum z_i,$$

here, the conditions of the problem imply that all the x_i are different from zero and have a sign coinciding with that of x (the same is true for y_i and z_i). Obviously, $|OA| \leqslant a$. On the other hand,

$$|x| + |y| + |z| = \sum (|x_i| + |y_i| + |z_i|)$$

$$\geqslant \sum l_i = a$$

(l_i is the length of the ith section of the polygonal line).

It can be readily shown that any point A satisfying the conditions $|OA| \leqslant a$, $|x| + |y| + |z| > a$, where x, y, z are the coordinates of the point A, can be the end point of a polygonal line consisting of not more than three sections and satisfying the conditions of the problem. Let, for instance, M_1 and M_2 be two points lying on one straight line emanating from the point O such that $|x_1| + |y_1| + |z_1| = a$, $x_1y_1z_1 \neq 0$ (x_1, y_1, z_1 the coordinates of the point M_1), $|OM_2| = a$. Consider the polygonal line with vertices $(0, 0, 0)$, $(x_1, 0, 0)$, $(x_1, y_1, 0)$, (x_1, y_1, z_1). The length of this polygonal line is equal to a. "Stretching" this line, we get all points of the line segment M_1M_2 (excluding M_1). Thus, the desired locus of points consists of all points lying outside the octahedron $|x| + |y| + |z| = a$ and inside or on the surface of the sphere with centre O and radius a. In this case, the points situated in the coordinate planes are excluded.

288. First of all note that if r is the radius of the ball inscribed in $ABCD$, then, firstly, all the edges of the tetrahedron $ABCD$ are longer than $2r$ and, secondly, the radius of the circle inscribed in any face of the tetrahedron is greater than r. The first assertion is obvious. To prove the second assertion, through the centre of the inscribed ball, pass a plane parallel, say, to the face ABC. The section cut is a triangle $A_1B_1C_1$ similar to the triangle ABC with the ratio of similitude less than unity and containing inside itself a circle of radius r.

(1) The condition determining the set of points A will be expressed by the inequality $|OA| \geqslant 3r$, the

equality $|OA| = 3r$ being true for a regular tetrahedron. If for some point A the inequality $|OA| < 3r$ were fulfilled, then the radius of the smallest ball containing the tetrahedron $ABCD$ would be less than $3r$, which is impossible (see Problem 246).

(2) The condition determining the set of points B will be expressed by the inequality $|OB| > r\sqrt{5}$. Indeed, if for some point B the inequality $|OB| \leqslant r\sqrt{5}$ were fulfilled, then for the triangle DBC the radius of the circle containing this triangle would be not greater than $\sqrt{5r^2 - r^2} = 2r$, that is, the radius of the circle inscribed in the triangle DBC would not exceed r, which is impossible.

(3) The condition determining the set of points C is expressed by the inequality $|OC| > r\sqrt{2}$. Indeed, if $|OC| \leqslant r\sqrt{2}$, then $|CD| \leqslant 2r$.

(4) The condition determining the set of points D will be expressed by the inequality $|OD| > r$.

Let us show that $|OD|$ can be arbitrarily large. To this effect, for the tetrahedron $ABCD$ take a tetrahedron all faces of which are congruent isosceles triangles having sufficiently small vertex angles. Then the centres of the inscribed and circumscribed balls will coincide, and the ratio $\frac{R}{r}$, where R is the radius of the circumscribed ball, can be arbitrarily large.

289. If MC is the hypotenuse of the appropriate triangle, then the equality $|MC|^2 = |MA|^2 + |MB|^2$ must be fulfilled. Introducing a rectangular Cartesian coordinate system, it is easy to make sure that the point M must describe the surface of a sphere. Find the centre and radius of this sphere.

Let C_1 be the midpoint of AB, C_2 lie on the extension of CC_1, $|C_1C_2| = |CC_1|$ ($ACBC_2$ is a parallelogram). Denote the sides of the triangle ABC, as usual, by a, b, and c, the median to the side AB by m_c. We shall have

$$|MA|^2 + |MB|^2 = 2|MC_1|^2 + \frac{|AB|^2}{2} = 2|MC_1|^2 + \frac{c^2}{2}.$$

Since

$$|MA|^2 + |MB|^2 = |MC|^2,$$

we get

$$|MC|^2-2|MC_1|^2=\frac{c^2}{2}. \qquad (1)$$

Let $\widehat{MC_2C}=\varphi$, write for the triangles MC_2C and MC_2C_1 the theorem of cosines:

$$|MC|^2=|MC_2|^2+4m_c^2-4|MC_2|m_c\cos\varphi, \qquad (2)$$

$$|MC_1|^2=|MC_2|^2+m_c^2-2|MC_2|m_c\cos\varphi. \qquad (3)$$

Multiplying (3) by 2 and subtracting the result from (2), we get (taking into account (1))

$$|MC_2|^2=2m_c^2-\frac{c^2}{2}=a^2+b^2-c^2.$$

Thus, for this case the set of points M will be non-empty if $a^2+b^2-c^2\geqslant 0$, that is, the angle C in the triangle ABC is not obtuse. Consequently, the whole set of points M for an acute-angled triangle consists of three spheres whose centres are found at the points C_2, A_2 and B_2 such that CAC_2B, ABA_2C, BCB_2A are parallelograms, the radii being respectively equal to $\sqrt{a^2+b^2-c^2}$, $\sqrt{b^2+c^2-a^2}$, and $\sqrt{a^2+c^2-b^2}$. For the right-angled triangle ABC the sought-for set consists of two spheres and a point, and for an obtuse-angled triangle of two spheres.

290. Let O denote the centre of the Earth, A the point on the equator corresponding to zero meridian, M the point on the surface of the Earth with longitude and latitude equal to φ, N the projection of M on the plane of the equator. Introducing a rectangular Cartesian coordinate system in the plane of the equator, taking the line OA for the x-axis, and the origin at the point O, we get that N has the following coordinates: $x=R\cos^2\varphi$, $y=R\cos\varphi\sin\varphi$, where R is the radius of the Earth. It is easy to check that the coordinates of the point N satisfy the equation

$$\left(x-\frac{R}{2}\right)^2+y^2=\frac{R^2}{4},$$

i.e. the sought-for set is a circle with centre $\left(\frac{R}{2}, 0\right)$ and radius $R/2$.

291. Introduce the following notation: S is the vertex of the cone, N the projection of the point M on the plane passing through the points S and A parallel to the base of the cone, P a point on the straight line SN such that $\widehat{SMP} = 90°$ (Fig. 56), MP is a normal to the surface

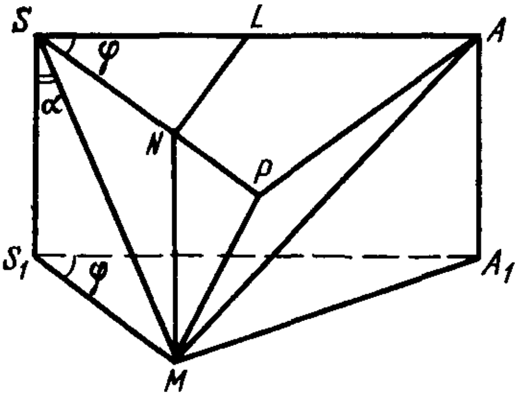


Fig. 56

of the cone. It follows from the hypothesis that AP is parallel to the reflected ray. Hence $\widehat{AMP} = \widehat{MPA}$, $|AM| = |AP|$. Let α be the angle between the altitude and generatrix of the cone $|SA| = a$. The plane passing through M parallel to the plane SPA cuts the axis of the cone at the point S_1, A_1 is the projection of A on this plane,

$$|SS_1| = x, \quad \widehat{MS_1A_1} = \varphi, \quad |MA_1| = y.$$

By the theorem of cosines for the triangle S_1MA_1, we have

$$y^2 = x^2 \tan^2 \alpha + a^2 - 2ax \tan \alpha \cos \varphi. \tag{1}$$

Besides,

$$|PA|^2 = |MA|^2 = y^2 + x^2, \tag{2}$$

$$|SP| = \frac{|SM|}{\sin \alpha} = \frac{x}{\cos \alpha \sin \alpha} = \frac{2x}{\sin 2\alpha}. \tag{3}$$

Writing the theorem of cosines for the triangle SPA and using the above relationships, we have

$$x^2 \tan^2 \alpha - 2ax \tan \alpha \cos \varphi + x^2 = \frac{4x^2}{\sin^2 2\alpha} - \frac{4ax}{\sin 2\alpha} \cos \varphi,$$

whence $x = a \sin 2\alpha \cos \varphi$.

If now we erect a perpendicular to SN at the point N in the plane SPA and denote by L the point of its intersection with SA, then

$$|SL| = \frac{|SN|}{\cos \varphi} = \frac{x \tan \alpha}{\cos \varphi} = 2a \sin^2 \alpha.$$

Thus, $|SL|$ is constant, consequently, the point N describes a circle with diameter SL.

292. When solving this problem, we shall need the following statements from plane geometry.

If in a circle of radius R through a point P found at a distance d from its centre two mutually perpendicular chords AD and BE are drawn, then

(a) $|AD|^2 + |BE|^2 = 8R^2 - 4d^2$,

(b) the perpendicular dropped from P on AB bisects the chord DE.

For a three-dimensional case, these two statements are generalized in the following way.

If through a point P found inside a ball of radius R centred at O three mutually perpendicular chords AD, BE, and CF are drawn at a distance d from its centre, then

(a*) $|AD|^2 + |BE|^2 + |CF|^2 = 12R^2 - 8d^2$,

(b*) a straight line passing through P perpendicular to the plane ABC passes through the median point of the triangle DEF.

Let us prove Item (a*). Let R_1, R_2, R_3 denote the radii of the circles circumscribed respectively about the quadrilaterals $ABDE$, $ACDF$, and $BCEF$, d_1, d_2, d_3 the distances in these quadrilaterals from the centres of the circumscribed circles to the point P, and x, y, z the respective distances from the point O to the planes of these quadrilaterals. Then $x^2 + y^2 + z^2 = d^2$, $d_1^2 + d_2^2 + d_3^2 = 2(x^2 + y^2 + z^2) = 2d^2$, $R_1^2 + R_2^2 + R_3^2 = 3R^2 - d^2$.

Thus, taking advantage of the statement of Item (a), we get

$$| AD |^2+| BE |^2+| CF |^2=\frac{1}{2}[(| AD |^2+| BE |^2)$$
$$+(| BE |^2+| CF |^2)+(| CF |^2+| AD |^2)]$$
$$=\frac{1}{2}(8R_1^2-4d_1^2+8R_1^2-4d_2^2+8R_3^2-4d_3^2)$$
$$=12R^2-8d^2.$$

To prove Item (b*), project the drawn line on the planes of the quadrilaterals $ABDE$, $ACDF$, and $BCEF$, and then take advantage of Item (b).

Now, let us pass to the statement of our problem. On the line segments PA, PB, and PC construct a parallelepiped and denote by M the vertex of this parallelepiped opposite the point P.

Analogously, determine the point N for the line segments PD, PE, and PF. K is the point of intersection of PM with the plane ABC, Q the midpoint of PM, T the midpoint of PN, O_1 the centre of the circle circumscribed about the triangle ABC, and H the foot of the perpendicular dropped from P on ABC.

It follows from Item (b*) that H lies on the straight line NP. Further, K is the point of intersection of the medians of the triangle ABC, $| PK | = \frac{1}{3} | PM |$. The straight line OQ is perpendicular to the plane ABC and passes through the point O_1, since O and Q are the centres of two spheres passing through the points A, B, and C. (Note that we have proved simultaneously that the points O_1, K, and H are collinear and $| KH | = 2 | O_1K |$. As is known, this straight line is called the *Euler line*.)

Thus, OQ is parallel to NP, the same as TO is parallel to MP. Hence, O is the midpoint of NM.

On the line segment OP take a point S such that $| PS | = \frac{1}{3} | PO |$. The perpendicular dropped from S on KH passes through the midpoint of KH. Consequently, $| SK | = | SH |$. But $SK \parallel OM$,

$$| SK | = \frac{1}{3} | OM | = \frac{1}{6} | NM |.$$

It follows from Item (a*) that $|NM|^2=12R^2-8d^2$ (NM is the diagonal of the parallelepiped whose edges are equal to $|AD|$, $|BE|$, $|CF|$), that is $|SK|=\frac{1}{3}\sqrt{3R^2-2d^2}$ is a quantity independent of the way in which the line segments PA, PB, PC were drawn.

293. Denote by **a**, **b**, and **c** the unit vectors directed along the edges of the trihedral angle, let, further, $\overrightarrow{ON}=\mathbf{e}$, P the centre of the sphere, $\overrightarrow{OP}=\mathbf{u}$, $\overrightarrow{OA}=x\mathbf{a}$, $\overrightarrow{OB}=y\mathbf{b}$, $\overrightarrow{OC}=z\mathbf{c}$.

The points O, N, A, B, and C belong to one and the same sphere with centre at P. This means that

$$(\mathbf{u}-\mathbf{e})^2=\mathbf{u}^2,\quad (x\mathbf{a}-\mathbf{u})^2=\mathbf{u}^2,$$
$$(y\mathbf{b}-\mathbf{u})^2=\mathbf{u}^2,\quad (z\mathbf{c}-\mathbf{u})^2=\mathbf{u}^2,$$

whence

$$\begin{cases} \mathbf{e}^2-2\mathbf{eu}=0, \\ x-2\mathbf{au}=0, \\ y-2\mathbf{bu}=0, \\ z-2\mathbf{cu}=0. \end{cases} \tag{1}$$

Let $\mathbf{e}=\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}$. Multiplying the second, third, and fourth equations of System (1) respectively by α, β, and γ and subtracting from the first, we obtain

$$\mathbf{e}^2-\alpha x-\beta y-\gamma z=0. \tag{2}$$

If M is the centre of gravity of the triangle ABC, then

$$\overrightarrow{OM}=\frac{1}{5}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})=\frac{1}{3}(x\mathbf{a}+y\mathbf{b}+z\mathbf{c}).$$

Taking into consideration Equation (2), we may conclude that the locus of points M is a plane.

294. Prove that each of these planes passes through the point symmetric to the point N with respect to the centre of gravity of the tetrahedron.

295. Prove that all these planes pass through the point symmetric to the centre of the sphere circumscribed

about the tetrahedron with respect to its centre of gravity.

296. When solving Problem 295, we proved that Monge's point is symmetric to the centre of the sphere circumscribed about the tetrahedron with respect to the centre of gravity of the tetrahedron. Consequently, if Monge's point belongs to the plane of some face of the tetrahedron, then the centre of the circumscribed sphere is situated from this face at a distance equal to half the length of the corresponding altitude and is located on the same side of the face on which the tetrahedron itself lies. This readily leads to the statement of our problem.

297. Take advantage of the equality

$$|MA|^2+|MB|^2=\frac{4|MD|^2+|AB|^2}{2},$$

where D is the midpoint of AB, and also by the fact that in an arbitrary tetrahedron the sum of the squares of its opposite edges is equal to twice the sum of the squares of the distances between the midpoints of two pairs of its remaining edges (see Problem 21).

298. Denote the areas of the faces of the tetrahedron by S_1, S_2, S_3, S_4 and the volume of the tetrahedron by V. If r is the radius of the sphere touching all the planes forming the tetrahedron, then, with the signs of $\varepsilon_i = \pm 1$, $i = 1, 2, 3, 4$, properly chosen, the equality $(\varepsilon_1 S_1 + \varepsilon_2 S_2 + \varepsilon_3 S_3 + \varepsilon_4 S_4)\frac{r}{3} = V$ must be fulfilled. In this case if for a given set ε_i the value of r determined by the last equality is positive, then the corresponding ball exists.

Thus, in an arbitrary tetrahedron there always exists one inscribed ball ($\varepsilon_i = +1$) and four externally inscribed balls (one $\varepsilon_i = -1$, the remaining ones $+1$), that is, four such balls each of which has the centre outside the tetrahedron and touches one of its faces at an interior point of this face.

Further, obviously, if for some choice of ε_i there exists a ball, then for an opposite set ε_i there exists no ball. This means that there are at most eight balls. There will be exactly eight balls if the sum of the areas of any two faces is not equal to the sum of the areas of two others.

299. For any two neighbouring sides of the quadrilateral there are two planes equidistant from them (the bisector planes of the angle of the quadrilateral itself and the angle adjacent to it). In this case, if three such planes corresponding to three vertices of the quadrilateral intersect at a certain point, then through this point there passes one of the two bisector planes of the fourth vertex. Thus, when finding the points equidistant from the lines forming the quadrilateral, it suffices to consider the bisector planes of three angles of this quadrilateral. Since two planes correspond to each vertex, there will be, generally speaking, eight points of intersection.

It remains to find out under what conditions some three such planes do not intersect. Since our quadrilateral is three-dimensional, no two bisector planes are parallel. Hence, there remains the possibility of one bisector plane to be parallel to the line of intersection of two others. And this means that if, through some point in space, three planes are passed parallel to the given ones, then these three planes will intersect along a straight line.

Let, for the sake of definiteness, the bisector planes of the three interior angles of the quadrilateral $ABCD$

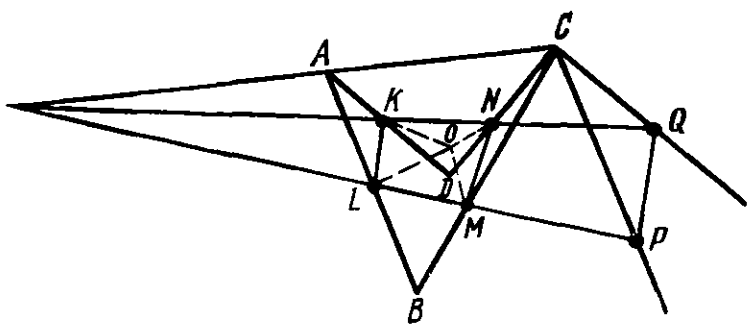


Fig. 57

not intersect. Through the vertex C, draw straight lines parallel to the sides AB and AD (Fig. 57) and on these lines lay off line segments CP and CQ, $|CP| = |CQ|$. Lay off equal line segments CM and CN on the sides CB and CD.

The aforegoing reasoning imply that the bisector planes of the angles MCP, PCQ, QCN, and NCM intersect along a straight line and, hence, all the points of this line are equidistant from the straight lines CP, CQ,

CN, CM, that is, the lines CP, CQ, CN, and CM lie on the surface of the cone, and $PQNM$ is an inscribed quadrilateral. Let the plane of the quadrilateral $PQNM$ intersect AB and AD at points L and K. The line LK is paralle to QP, and this means that $NMLK$ is also an inscribed quadrilateral. Besides, it is easily seen that

$$|LB| = |MB|, \quad |KD| = |DN|, \quad |KA| = |AL|.$$

Hence, in particular, it follows that $|AB| + |DC| = |AD| + |BC|$.

Let now O denote the centre of the circle circumscribed about the quadrilateral $KLMN$. The congruence of the triangles LOB and MOB implies that O is equidistant from the lines AB and BC. Proceeding in the same way, we will show that O is equidistant from all the lines forming the quadrilateral $ABCD$, that is, O is the centre of the ball touching the straight lines AB, BC, CD, and DA. Other cases are considered exactly in the same manner to obtain analogous relationships among the sides of $ABCD$: $|AB| + |AD| = |CD| + |CB|$, $|AB| + |BC| = |AD| + |DC|$. It is not difficult to show that the indicated relationships among the sides of the quadrilateral $ABCD$ are the necessary and sufficient conditions for the existence of infinitely many balls touching the sides of the quadrilateral. In all remaining cases there are exactly eight such balls.

300. Using the formula of Problem 11 for the volume of the tetrahedron, prove that each of the relationships under consideration is equal to $\frac{4S_1S_2S_3S_4}{9V^2}$, where S_1, S_2, S_3, S_4 are the areas of the faces of the tetrahedron, V its volume.

301. If h_i ($i = 1, 2, 3, 4$) is the altitude of the corresponding face of the tetrahedron, then

$$\frac{1}{3}\sqrt{\frac{1}{2}\sum_{i=1}^{4} S_i^2(l_i^2 - R_i^2)} = \frac{1}{3}\sqrt{\frac{1}{2}\sum_{i=1}^{4} S_i^2 h_i^2 \frac{l_i^2 - R_i^2}{h_i^2}}$$

$$= V\sqrt{\frac{1}{2}\sum_{i=1}^{4} \frac{l_i^2 - R_i^2}{h_i^2}}.$$

If now d_i is the distance from the centre of the circumscribed ball to the ith face (R is the radius of this ball), then

$$l_i^2 - R_i^2 = (l_i^2 - h_i^2) - (R^2 - d_i^2) + h_i^2$$
$$= [R^2 - (h_i - d_i)^2] - (R^2 - d_i^2) + h_i^2 = 2h_i d_i.$$

Thus, we get the following radicand:

$$\sum \frac{d_i}{h_i} = 1$$

(see Problem 182), which was required to be proved. (We assumed that the centre of the circumscribed ball lies inside the tetrahedron. If the centre is found outside it, proceed in the same way regarding one of the quantities d_i as being negative.)

302. Denote the lengths of the edges of the tetrahedron $ABCD$ as is shown in Fig. 58, *a*. Through the vertex A pass a plane tangent to the ball circumscribed about the tetrahedron $ABCD$. The tetrahedron ABC_1D_1 in this figure is formed by this tangent plane, the planes ABC, ABD, and also by the plane passing through B parallel to the face ADC. Analogously, the tetrahedron AB_2C_2D is formed by the same tangent plane, the planes ABD, ADC, and the plane passing through D parallel to ABC.

From the similarity of the triangles ABC and ABC_1 (Fig. 58, *b*, AC_1 is a tangent line to the circle circumscribed about the triangle ABC, consequently, $\widehat{BAC_1} = \widehat{BCA}$, besides, $BC_1 \parallel AC$, hence, $\widehat{C_1BA} = \widehat{BAC}$) find $|AC_1| = \frac{ac}{b}$. Analogously, find $|AD_1| = \frac{nc}{m}$, $|AC_2| = \frac{mp}{b}$, $|AB_2| = \frac{mn}{c}$. But the triangles AC_1D_1 and AB_2C_2 are similar, hence

$$\frac{|C_1D_1|}{|AC_2|} = \frac{|AD_1|}{|AB_2|}, \quad |C_1D_1| = \frac{pc^2}{bm}.$$

Note that if the lengths of the sides of the triangle are multiplied by $\frac{bm}{c}$, then these lengths will turn out to be numerically equal to the quantities am, bn, and cp, thus

$$S_{AD_1C_1}=\frac{c^2}{b^2m^2}S.$$

Let, further, AM denote the diameter of the circumscribed ball and BK the altitude of the pyramid

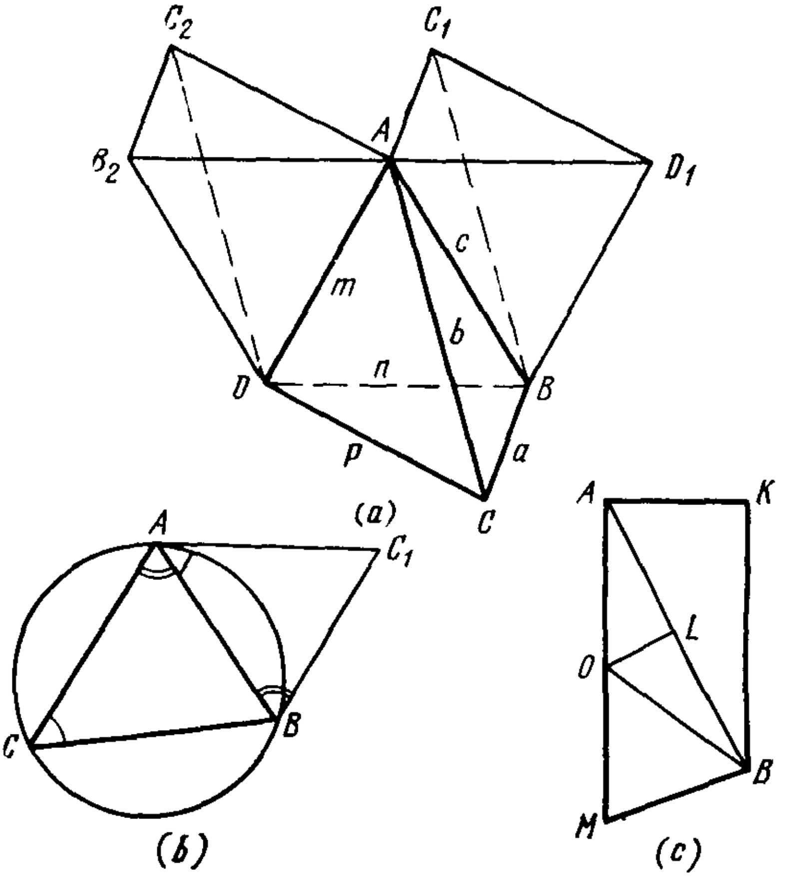


Fig. 58

ABC_1D_1 dropped from B on AC_1D_1 (Fig. 58, c). From the similarity of the triangles ABK and OLA (OL is perpendicular to AB) we find $|BK|=\frac{c^2}{2R}$. Hence,

$$V_{AD_1C_1B}=\frac{1}{3}\,\frac{c^4}{2Rb^2m^2}S.$$

And, finally,

$$\frac{V_{AD_1C_1B}}{V}=\frac{S_{ABC_1}S_{ABD_1}}{S_{ABC}S_{ABD}}=\frac{c^2}{b^2}\cdot\frac{c^2}{m^2}, \quad V_{AD_1C_1B}=\frac{c^4}{b^2m^2}V.$$

Comparing two expressions for $V_{AD_1C_1B}$, we get the truth of the statement in question.

Remark. It follows from our reasoning that the angles of the triangle the lengths of the sides of which are numerically equal to the products of the lengths of the opposite edges of the tetrahedron are equal to the angles between the tangents to the circles circumscribed about three faces of the tetrahedron. The tangents are drawn through the vertex common for these faces and are situated in the plane of the appropriate face. It is readily seen, that the same will also be true for a degenerate tetrahedron, that is, for a plane quadrilateral. Hence, in particular, it is possible to obtain the theorem of cosines (Bretschneider's theorem, see p. 171) for a plane quadrilateral.

303. Let S_1 and S_2 denote the areas of the faces having a common edge a, S_3, and S_4 the areas of the two remaining faces. Let, further, a, m, and n denote the lengths of the edges forming the face S_1, and α, γ, and δ the dihedral angles adjacent to them, V the volume of the tetrahedron. Then it is readily verified that the following equality is true:

$$a\,\frac{3V}{S_1}\cot\alpha+m\,\frac{3V}{S_1}\cot\gamma+n\,\frac{3V}{S_1}\cot\delta=2S_1,$$

or

$$a\cot\alpha+m\cot\gamma+n\cot\delta=\frac{2S_1^2}{3V}.$$

Writing such equalities for all the faces of the tetrahedron, adding together the equalities corresponding to the faces S_1 and S_2, and subtracting the two others, we get

$$a\cot\alpha-b\cot\beta=\frac{1}{3V}\left(S_1^2+S_2^2-S_3^2-S_4^2\right).$$

Squaring this equality, replacing $\cot^2\alpha$ and $\cot^2\beta$ by $\frac{1}{\sin^2\alpha}-1$ and $\frac{1}{\sin^2\beta}-1$, and taking advantage of the following equalities:

$$\frac{a^2}{\sin^2\alpha}=\frac{4S_1^2S_2^2}{9V^2}, \quad \frac{b^2}{\sin^2\beta}=\frac{4S_3^2S_4^2}{9V^2}$$

(see Problem 11), we finally get

$$a^2+b^2+2ab\cot\alpha\cot\beta=\frac{1}{9V^2}(2Q-T),$$

with Q the sum of the squares of the pairwise products of the areas of the faces, and T the sum of the fourth powers of the areas of the faces.

304. The necessity of all conditions is obvious. We are going to prove their sufficiency.

(a) The statement of the problem is readily proved by making the development of the tetrahedron (to this end, the surface of the tetrahedron should be cut along three edges emanating from one vertex).

(b) Make the development of the tetrahedron $ABCD$ following Fig. 59, a in the supposition that the sums of

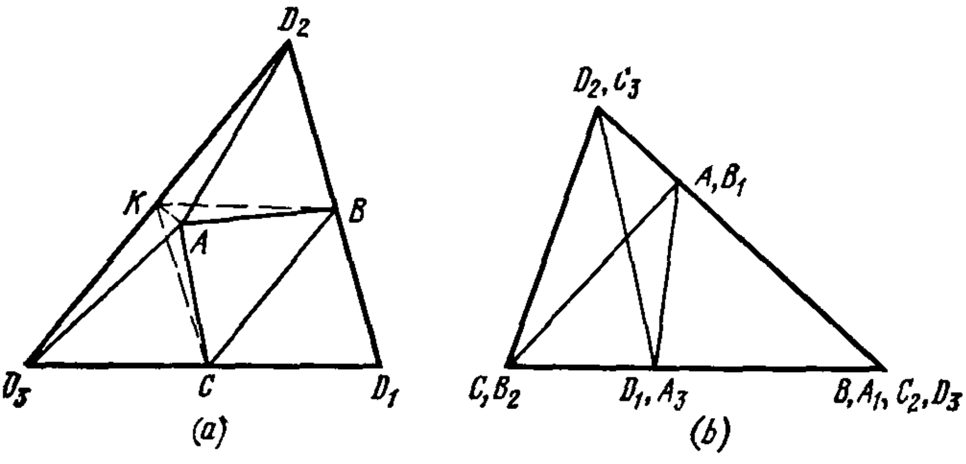


Fig. 59

the plane angles at the vertices B and C are equal to 180°. The points D_1, D_2, and D_3 correspond to the vertex D. Two cases are possible:

(1) $|AD|=|BC|$. In this case $|D_3A|+|D_2A|=2|BC|=|D_3D_2|$, that is, the triangle D_2AD_3 degenerates, the point A must coincide with the point K which is the midpoint of D_2D_3.

(2) $|AB| = |CD|$ (or $|AC| = |BD|$). In this case $|KB| = |AB|$, the point A being found on the middle perpendicular to the side D_2D_3. If $D_1D_2D_3$ is an acute-angled triangle, then $|AB| < |KB|$ for points A situated inside the triangle KBC, and $|AB| > |KB|$ for the points situated outside the triangle KBC.

And if the triangle $D_1D_2D_3$ is obtuse-angled (an obtuse angle being either at the vertex D_2 or at the vertex D_3), then at one of the two vertices of the tetrahedron (either B or C) one plane angle will be greater than the sum of two other angles.

(c) Let $|AB| = |CD|$, $|AC| = |DB|$, and the sum of the angles at the vertex D is equal to 180°. We have: the triangle ACD is congruent to the triangle ABD, consequently, $\widehat{ADB} = \widehat{DAC}$.

Thus $\widehat{ADB} + \widehat{ADC} + \widehat{CDB} = \widehat{DAC} + \widehat{ADC} + \widehat{CDB} = 180°$. Hence, it follows that $\widehat{CDB} = \widehat{ACD}$ and $\triangle ACD = \triangle CDB$, $|AD| = |CB|$.

(d) Cut the tetrahedron along the edges, and superimpose the four triangles thus obtained one over another so as to bring to coincidence their equal angles. In Fig. 59, *b*, identical letters correspond to one and the same vertex of the tetrahedron, and identical subscripts to one and the same face. Identical letters corresponding to one point show that at this point the corresponding vertices of the appropriate triangles coincide. Consequently,

$$|C_3A_3| = |CA|, \quad |B_2D_2| = |B_1D_1|$$

and this means that AC_3 is parallel to B_2D_1 which is impossible.

(e) Project the tetrahedron $ABCD$ on the plane parallel to the edges AB and CD. Then it is possible to prove that the projections of the triangles ABC and ABD will be equivalent. Exactly in the same manner, the projections of the triangles ACD and BCD will also be equivalent. And this means that the parallelogram with diagonals AB and CD will be the projection of $ABCD$. Hence follow the equalities $|AC| = |BD|$, $|AD| = |BC|$. The equality $|AB| = |CD|$ is proved exactly in the same way.

(f) Let O_1 denote the point of tangency of the inscribed sphere with the face ABC, and O_2 with the face BCD. The hypothesis implies that O_1 and O_2 are the centres of the circles circumscribed about ABC and BCD. Besides, the triangle BCO_1 is congruent to the triangle BCO_2 This implies that

$$\widehat{BAC} = \frac{1}{2}\widehat{BO_1C} = \frac{1}{2}\widehat{BO_2C} = \widehat{BDC}.$$

Reasoning in the same way, we shall obtain that all the plane angles adjacent to the vertex D are equal to the corresponding angles of the triangle ABC, that is, their sum is equal to 180°. The same may be asserted about the remaining vertices of the tetrahedron $ABCD$. Further, take advantage of Item (a).

(g) Complete the given tetrahedron to get a parallelepiped in a usual way, that is, by passing through each edge of the tetrahedron a plane parallel to the opposite edge. Then the necessary and sufficient condition of the equality of the faces of the tetrahedron will be expressed by the condition that the obtained parallelepiped be rectangular. And from the fact that the edges of this parallelepiped are equal and parallel to the corresponding line segments joining the midpoints of opposite edges of the tetrahedron will follow our statement.

(h) If O is the centre of the sphere circumscribed about the tetrahedron $ABCD$, then the hypothesis will imply that the triangle AOB is congruent to the triangle COD, since both triangles are isosceles with equal lateral sides, equal medians emanating from the vertex O (O coincides with the midpoint of the line segment joining the midpoints of AB and CD). Consequently, $|AB| = |CD|$. The equality of other pairs of opposite edges is proved exactly in the same manner.

(i) From the fact that the distances from the centres of gravity to all the faces are equal follows the equality of the altitudes of the tetrahedron and then also the equality of its faces (see Item (e)).

305. Let **a**, **b**, **c**, and **d** denote vectors perpendicular to the faces of the tetrahedron, directed outside and having the length numerically equal to the area of the corresponding face, and let $\mathbf{e}_a$, $\mathbf{e}_b$, $\mathbf{e}_c$, and $\mathbf{e}_d$ denote the unit

vectors having the same directions as **a**, **b**, **c**, and **d**. Let, further, s denote the sum of the cosines of the dihedral angles, and $\mathbf{k} = \mathbf{e}_a + \mathbf{e}_b + \mathbf{e}_c + \mathbf{e}_d$.

It is obvious that $k^2 = 4 - 2s$. Thus, indeed, $s \leqslant 2$ and $s = 2$ if and only if $\mathbf{k} = \mathbf{e}_a + \mathbf{e}_b + \mathbf{e}_c + \mathbf{e}_d = 0$. But since $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0$ (see Problem 214), we obtain that for $s = 2$ the lengths of the vectors **a**, **b**, **c**, and **d** are equal to one another, i.e. all the faces are equivalent, and from the equivalency of the faces there follows their congruence (see Problem 304 (e)). To complete the proof, it remains to show that $s > 0$ or that $|\mathbf{k}| < 2$.

For conveniency, we shall regard that $|\mathbf{a}| = 1$, $|\mathbf{b}| \leqslant 1$, $|\mathbf{c}| \leqslant 1$, $|\mathbf{d}| \leqslant 1$. Then $\mathbf{e}_a = \mathbf{a}$, $|\mathbf{k}| = |\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} + (\mathbf{e}_b - \mathbf{b}) + (\mathbf{e}_c - \mathbf{c}) + (\mathbf{e}_d - \mathbf{d})| \leqslant |\mathbf{e}_b - \mathbf{b}| + |\mathbf{e}_c - \mathbf{c}| + |\mathbf{e}_d - \mathbf{d}| = 3 - (|\mathbf{b}| + |\mathbf{c}| + |\mathbf{d}|) \leqslant 3 - |\mathbf{b} + \mathbf{c} + \mathbf{d}| = 3 - |\mathbf{a}| = 2$. Equality may be the case only if all the vectors **a**, **b**, **c**, and **d** are collinear; since it is not so, $|\mathbf{k}| < 2$, $s > 0$.

306. Consider the tetrahedron all faces of which are congruent triangles whose angles are respectively equal to the plane angles of our trihedral angle. (Prove that such tetrahedron exists.) All the trihedral angles of this tetrahedron are equal to the given trihedral angle. The sum of the cosines of the dihedral angles of such tetrahedron is equal to 2 (see Problem 304). Consequently, the sum of the cosines of the dihedral angles of the given trihedral angle is equal to 1.

307. Constructing a parallelepiped from the given tetrahedron, and passing through each edge a plane parallel to the opposite edge, we shall get for the equifaced tetrahedron, as is known, a rectangular parallelepiped.

The centre of the inscribed ball coincides with the centre of the parallelepiped, and the centres of the externally inscribed balls are found at the vertices of the parallelepiped different from the vertices of the tetrahedron. This implies both statements of the problem.

308. Let $ABCD$ be the given tetrahedron, DH its altitude, DA_1, DB_1, and DC_1 the altitudes of the faces dropped from the vertex D on the sides BC, CA, and AB. Cut the surface of the tetrahedron along the edges DA, DB, and DC and make the development (Fig. 60). It is obvious that H is the point of intersection of the altitudes of the triangle $D_1D_2D_3$. Let F denote the point of intersection of the altitudes of the triangle ABC, AK the

altitude of this triangle, $|AF| = h_1$, $|FK| = h_2$. Then $|D_1H| = 2h_1$, $|D_1A_1| = h_1 + h_2$, $|HA_1| = |h_1 - h_2|$. Hence, since h is the altitude of our tetrahedron,

$$h^2 = |DH|^2 = |DA_1|^2 - |HA_1|^2$$
$$= (h_1 + h_2)^2 - (h_1 - h_2)^2 = 4h_1h_2.$$

Now, let M denote the centre of gravity of the triangle ABC (it also serves as the centre of gravity of the triangle $D_1D_2D_3$), O the centre of the circle circumscribed

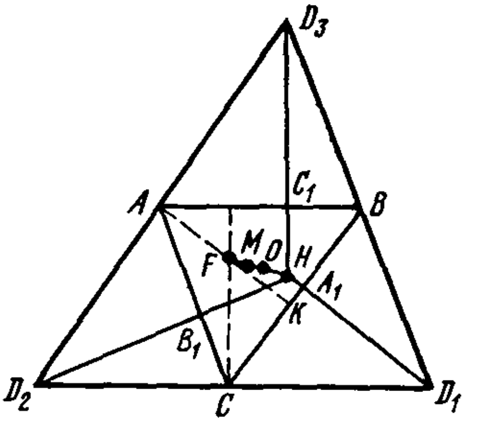


Fig. 60

about this triangle. It is known that F, M, and O lie on one and the same straight line (*Euler's line*), M lying between F and O, $|FM| = 2|MO|$.

On the other hand, the triangle $D_1D_2D_3$ is homothetic to the triangle ABC with centre at M and ratio of similitude equal to (-2), hence, $|MN| = 2|FM|$. Hence it follows that $|OH| = |FO|$.

309. When solving the preceding problem, we proved that the centre of the sphere circumscribed about the tetrahedron is projected on each edge into the midpoint of the line segment whose end points are the foot of the altitude dropped on this face and the point of intersection of the altitudes of this face. And since the distance from the centre of the sphere circumscribed about the tetrahedron to the face is equal to $\frac{1}{4}h$, where h is the altitude of the tetrahedron, the centre of the circumscribed sphere

is found at a distance of $\sqrt{\frac{1}{16}h^2+a^2}$ from the given points, where a is the distance between the point of intersection of the altitudes and the centre of the circle circumscribed about the face.

310. First of all, let us note that all the triangles ABC are acute. Indeed, if H is the point of intersection of the altitudes of the triangle ABC, O the centre of the given circle, then $|OH| = 3|OM|$, M lying between O

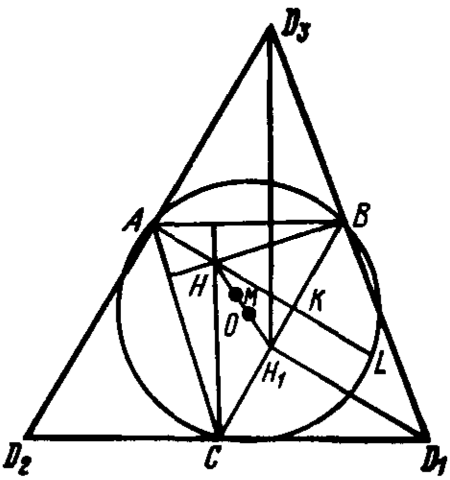


Fig. 61

and H, that is, H is found inside the circle circumscribed about the triangle ABC, and this means that the triangle ABC is acute, consequently, there is a point D such that $ABCD$ is an equifaced tetrahedron. Let us develop this tetrahedron (Fig. 61). Obviously, H_1, which is the point of intersection of the altitudes of the triangle $D_1D_2D_3$, is the foot of the altitude dropped from D on ABC. But the triangles ABC and $D_1D_2D_3$ have a common centre of gravity M with respect to which they are homothetic with the ratio of similitude (-2), hence $|H_1M| = 2|MH|$, M lying between H_1 and H, H_1 is a fixed point. It remains to prove that the altitude of the tetrahedron $ABCD$ is also constant. In the triangle ABC draw the altitude AK and extend it to intersect the circumscribed circle at point L. It is known (and is readily proved) that $|LK| = |KH|$. Let $|AH| = h_1$,

$|HK| = h_2$, the altitude of the tetrahedron is h. We know (see Problem 307) that $h^2 = 4h_1h_2 = 2|AH| \times |HL| = 2(R^2 - 9a^2)$, where $a = |OM|$, which was required to be proved.

311. Consider the cube $AEFGA_1E_1F_1G_1$ with edge equal to the side of the square $ABCD$. On the edges A_1E_1 and A_1G_1 take the points P and Q such that $|A_1P| = |BP| = |CQ|$, $|A_1Q| = |QD| = |PC|$ (Fig. 62, a). Con-

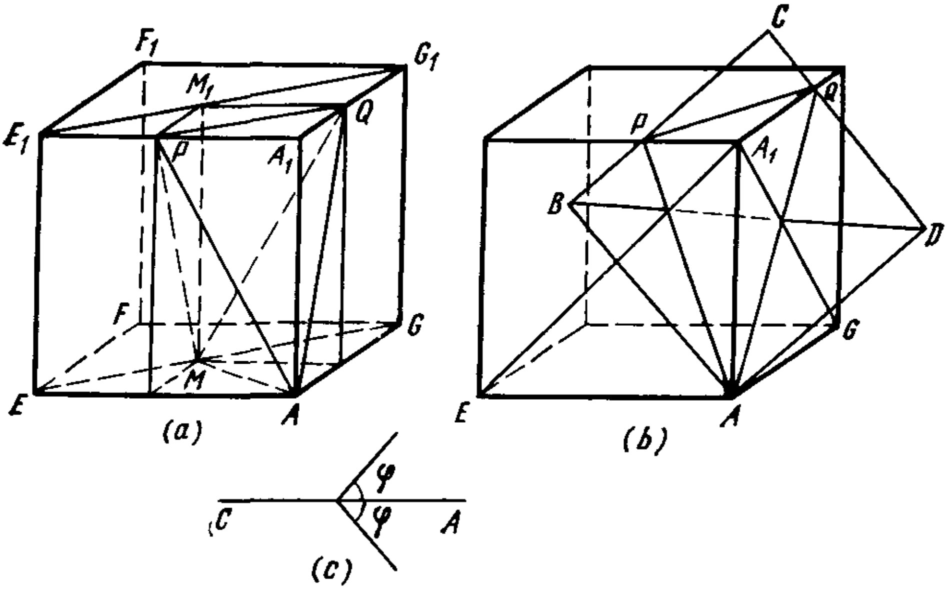


Fig. 62

sider the rectangle A_1PM_1Q. In view of the condition $|A_1P| + |A_1Q| = |A_1E_1|$, the point M_1 lies on the diagonal E_1G_1. Consequently, if M is the projection of M_1 on EG, then the tetrahedron $APQM$ has all the faces equal to the triangle APQ. The square $ABCD$ whose plane contains the triangle APQ is obtained from the square AEE_1A_1 by rotating about the diagonal AF_1 through some angle α (Fig. 62, b). Since the plane EGA_1 is perpendicular to the diagonal AF_1, BD belongs to this plane. But the planes AEE_1A_1, $ABCD$, as well as the straight lines EG, EA_1, A_1G, and BD are tangent to the ball inscribed in the cube. Hence it follows that the angle between the planes $ABCD$ and A_1EG has a constant size, it is equal to the angle φ between the planes AEE_1A_1

and A_1EG for which $\cos\varphi = \frac{1}{\sqrt{3}}$. But the planes A_1EG and $ABCD$ intersect along the diagonal BD. Hence, the point M lies in the plane passing through BD and making an angle φ with the plane $ABCD$, and the locus of projections of points M will be represented by two line segments emanating from the midpoint of AC at an angle φ to AC so that $\cos\varphi = \frac{1}{\sqrt{3}}$, and having the length $a\frac{\sqrt{2}}{2}$ (Fig. 62, c).

312. (a) Let $ABCD$ denote the given tetrahedron. If its altitudes intersect at the point H, then DH is perpendicular to the plane ABC and, hence, DH is perpendicular to BC. Exactly in the same way, AH is perpendicular to BC. Consequently, the plane DAH is perpendicular to BC, that is, the edges DA and BC are mutually perpendicular.

Conversely, let the opposite edges of the tetrahedron $ABCD$ be pairwise perpendicular. Through DA pass a plane perpendicular to BC. Let us show that the altitudes of the tetrahedron drawn from the vertices A and D lie in this plane.

Denote by K the point of intersection of the passed plane and the edge BC. The altitude DD_1 of the triangle ADK will be perpendicular to the lines AK and BC, hence, it is an altitude of the tetrahedron. Thus, any two altitudes of the tetrahedron intersect, hence, all the four intersect at one point.

(b) It is easy to prove that if one altitude of the tetrahedron passes through the point of intersection of the altitudes of the appropriate face, then the opposite edges of the tetrahedron are pairwise perpendicular. This follows from the theorem on three perpendiculars. Hence, Items (a) and (b) are equivalent.

(c) The equality of the sums of the squares of opposite edges of the tetrahedron is equivalent to the condition of the perpendicularity of opposite edges (see Item (a)).

(d) Complete the tetrahedron to a parallelepiped, as usual, by passing through each of its edges a plane parallel to the opposite edge. The edges of the obtained parallelepiped are equal to the distance between the midpoints of the skew edges of the tetrahedron. On the other hand, the condition of perpendicularity of opposite edges of the tetrahedron which is, according to Item (a), equivalent to the condition of the orthocentricity of the given tetrahedron, is, in turn, equivalent to the condition of the equality of the edges of the obtained parallelepiped (the diagonals of each face are equal and parallel to two opposite edges of the tetrahedron, that is, each face must be a rhombus).

(e) From Problems 300 and 303 it follows that this condition is equivalent to the condition of Item (c).

(f) Let a and a_1, b and b_1, c and c_1 be the lengths of three pairs of opposite edges of the tetrahedron, α the angle between them. From Problem 185 it follows that of the three numbers $aa_1 \cos\alpha$, $bb_1 \cos\alpha$, and $cc_1 \cos\alpha$ one is equal to the sum of two others. If $\cos\alpha \neq 0$, then of the three numbers aa_1, bb_1, and cc_1 one number is equal to the sum of two others. But this is impossible, since there is a triangle the lengths of the sides of which are numerically equal to the quantities aa_1, bb_1, and cc_1 (see Problem 302).

313. Let $ABCD$ denote the given tetrahedron. Complete it to get a parallelepiped in a usual way. Since

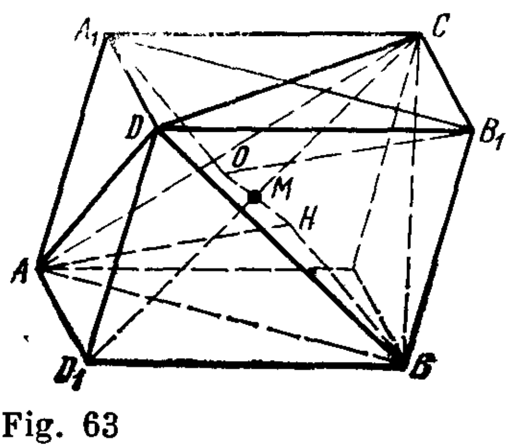


Fig. 63

$ABCD$ is an orthocentric tetrahedron, all the edges of the parallelepiped will be equal in length. Let A_1B_1 be the diagonal of a face of the parallelepiped parallel to AB, O the centre of the ball circumscribed about $ABCD$, H the point of intersection of the altitudes, M the centre of gravity (Fig. 63). Then the triangles ABH and A_1B_1O are symmetric with respect to the point M. This follows from the fact that ABB_1A_1 is a parallelogram and, besides, A_1O is perpendicular to the plane ACD (the points O and A_1 are equidistant from the points A, C, and D), and, hence, parallel to BH. Exactly in the same manner, OB_1 is parallel to AH.

314. Let us introduce the notation used in the preceding problem. Let K and L be the midpoints of AB and A_1B_1. Then $KOLH$ is a parallelogram. Consequently,

$$|OH|^2=2|OK|^2+2|OL|^2-|KL|^2$$
$$=2\left(R^2-\frac{|AB|^2}{4}\right)+2\left(R^2-\frac{|CD|^2}{4}\right)-l^2$$
$$=4R^2-\frac{1}{2}(|AB|^2+|CD|^2)-l^2=4R^2-3l^2.$$

315. If $ABCD$ is an orthocentric tetrahedron, then (see Problem 312 (d))

$$|AB|^2+|CD|^2=|AD|^2+|BC|^2,$$

whence

$$|AB|^2+|AC|^2-|BC|^2=|AD|^2+|AC|^2-|CD|^2,$$

that is, the angles $\widehat{BAC}$ and $\widehat{DAC}$ are both acute or obtuse.

316. The section of an orthocentric tetrahedron by any plane parallel to opposite edges and passing at an equal distance from these edges is a rectangle whose diagonals are equal to the distance between the midpoints of opposite edges of the tetrahedron (all these distances are equal in length, see Problem 312 (d)).

Hence it follows that the midpoints of all the edges of an orthocentric tetrahedron lie on the surface of the sphere whose centre coincides with the centre of gravity of the given tetrahedron and the diameter is equal to the distance between the opposite edges of the tetrahedron. Hence, all the four 9-point circles lie on the surface of this sphere.

317. Let O, M, and H respectively denote the centre of the circumscribed ball, centre of gravity and orthocentre (the point of intersection of altitudes) of the ortho-

centric tetrahedron, M the midpoint of the line segment OH (see Problem 313). The centres of gravity of the faces of the tetrahedron serve as the vertices of the tetrahedron, homothetic to the given one, with the centre of similitude at the point M and the ratio of similitude equal to $-(1/3)$. In this homothetic transformation the point O will move into the point O_1 situated on the line segment MH so that $|MO_1| = 1/3\,|OM|$, O_1 will be the centre of the sphere passing through the centres of gravity of the faces.

On the other hand, the points dividing the line segments of the altitudes of the tetrahedron from the vertices to the orthocentre in the ratio 2 : 1 serve as the vertices

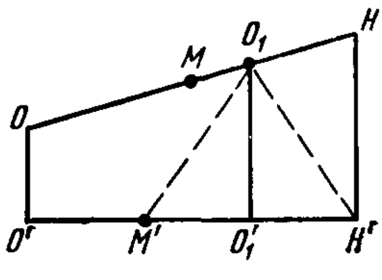


Fig. 64

of the tetrahedron homothetic to the given with the centre of similitude at H and the ratio of similitude equal to 1/3. In this homothetic transformation the point O, as is readily seen, will go to the same point O_1. Thus, eight of twelve points lie on the surface of the sphere with centre at O_1 and radius equal to one-third the radius of the sphere circumscribed about the tetrahedron.

Prove that the points of intersection of altitudes of each face lie on the surface of the same sphere. Let O', H', and M' denote, respectively, the centre of the circumscribed circle, the point of intersection of altitudes, and the centre of gravity of some face. O' and H' are the respective projections of O and H on the plane of this face, and the point M' divides the line segment $O'H'$ in the ratio 1 : 2 as measured from the point O' (a well-known fact from plane geometry). Now, we easily make sure (see Fig. 64) that the projection of O_1 on the plane of this face (point O_1') coincides with the midpoint of the line segment $M'H'$, that is, O_1 is equidistant from M' and H' which was required to be proved.

318. The centres of gravity of the faces of the orthocentric tetrahedron lie on the surface of the sphere homothetic to the sphere circumscribed about the tetrahedron with the centre of similitude at the point M and the ratio of similitude equal to 1/3 (see the solution of Problem 317). Hence follows the statement of the problem.

319. The feet of the altitudes of the orthocentric tetrahedron lie on the surface of the sphere homothetic to the sphere circumscribed about the tetrahedron with the centre of similitude at the point G and ratio of similitude equal to $-(1/3)$ (see the solution of Problem 317). Hence follows the statement of the problem.

320. Suppose the contrary. Let the planes containing the arcs intersect pairwise on the surface of the ball at points A and A_1, B and B_1, C and C_1 (Fig. 65). Since each

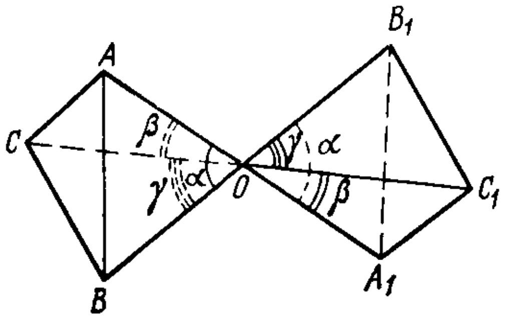


Fig. 65

arc measures more than 180°, it must contain at least one of any two opposite points of the circle on which it is situated. Let us enumerate these arcs and, respectively, the planes they lie in: I, II, III. A and A_1 are the points of intersection of planes I and II, B and B_1 the points of intersection of planes II and III, C and C_1 the points of intersection of planes III and I. Each of the points A, A_1, B, B_1, C, C_1 must belong to one arc. Let A_1 and C_1 belong to arc I, B_1 to arc II. Then B and C must belong to arc III, A to arc II. Denote by α, β, γ the plane angles of the trihedral angles, as is shown in the figure, O the centre of the sphere. Since arc I does not contain the points A and C, the inequality $360° - \beta > 300°$ must be fulfilled.

Similarly, since arc II does not contain the points B and A_1, it must be $180° + \alpha > 300°$ and, finally, for arc III we will have $360° - \gamma > 300°$. Thus, $\beta < 60°$, $\alpha > 120°$, $\gamma < 60°$, hence, $\alpha > \beta + \gamma$, which is impossible.

321. Let A and B denote two points on the surface of the sphere, C a point on the smaller arc of the great circle passing through A and B.

Prove that the shortest path from A to B must pass through C. Consider two circles α and β on the surface of the sphere passing through C with centres on the radii OA and OB (O the centre of the sphere). Let the line joining A to B does not pass through C and intersect the circle α at point M and the circle β at N.

Rotating the circle α together with the part of the line enclosed inside it so that M coincides with C and the circle β so as to bring N in coincidence with C, we get a line joining A and B whose length, obviously, is less than the length of the line under consideration.

322. The circumscribed sphere may not exist. It can be exemplified by the polyhedron constructed in the following way. Take a cube and on its faces as on bases construct outwards regular quadrangular pyramids with dihedral angles at the base equal to 45°. As a result, we get a dodecahedron (the edges of the cube do not serve as the edges of this polyhedron), having fourteen vertices, eight of which are the vertices of the cube, and six are the vertices of the constructed pyramids not coinciding with the vertices of the cube.

It is easy to see that all the edges of this polyhedron are equal in length and equidistant from the centre of the cube, while the vertices cannot belong to one sphere.

323. Let us note, first of all, that the area of the spherical lune formed by the intersection of the surface of the sphere with the faces of the dihedral angle of size α, whose edge passes through the centre of the sphere, is equal to $2\alpha R^2$. This follows from the fact that this area is proportional to the magnitude of α, and for $\alpha = \pi$ it is equal to $2\pi R^2$.

To each pair of planes forming the two faces of the given trihedral there correspond two lunes on the surface of the sphere. Adding their areas, we get the surface of the sphere enlarged by $4S_\Delta$, where S_Δ is the area of the desired triangle. Thus,

$$S_\Delta = R^2 (\alpha + \beta + \gamma - \pi).$$

The quantity $\alpha + \beta + \gamma - \pi$ is called the *spheric excess* of the spheric triangle.

324. Consider the sphere with centre inside the polyhedron and project the edges of the polyhedron from the centre of the sphere on its sphere.

The surface of the sphere will be broken into polygons. If n_k is the number of sides of the kth polygon, A_k the sum of its angles, S_k the area, then

$$S_k = R^2 [A_k - \pi (n_k - 2)].$$

Adding together these equalities for all K, we get

$$4\pi R^2 = R^2 (2\pi N - 2\pi k + 2\pi M).$$

Hence,

$$N - K + M = 2.$$

325. Let α denote the central angle corresponding to the spheric radius of the circle (the angle between the radii of the sphere drawn from the centre of the sphere to the centre of the circle and a point on the circle).

Consider the spheric triangle corresponding to the trihedral angle with vertex at the centre of the sphere one edge of which (OL) passes through the centre of the circle, another (OA), through the point on the circle, and a third (OB) is arranged so that the plane OAB touches the circle, the dihedral angle at the edge OL being equal to φ, $\widehat{LOA} = \alpha$.

Applying the second theorem of cosines (see Problem 166), find the dihedral angle at the edge OB, it is equal to arccos ($\cos \alpha \sin \varphi$). Any circumscribed polygon (our polygon can be regarded as circumscribed, since otherwise its area could be reduced) can be divided into triangles of the described type. Adding their areas, we shall see that the area of the polygon reaches the smallest value together with the sum $\arccos (\cos \alpha \sin \varphi_1) + \arccos (\cos \alpha \sin \varphi_2) + \ldots + \arccos (\cos \alpha \sin \varphi_N)$, where $\varphi_1, \ldots, \varphi_N$ are the corresponding dihedral angles, $\varphi_1 + \varphi_2 + \ldots + \varphi_N = 2\pi$. Then we can take advantage of the fact that the function $\arccos (k \sin \varphi)$ is a concave (or convex downward) function for $0 < k < 1$. Hence it follows that the minimum of our sum is reached for $\varphi_1 = \varphi_2 = \ldots = \varphi_N$.

326. Denote, as in Problem 324, by N the number of faces, by K the number of edges, and by M the number of vertices of our polyhedron,

$$N - K + M = 2. \tag{1}$$

Since from each vertex there emanate at least three edges and each edge is counted twice, $M \leqslant \frac{2}{3}K$. Substituting M into (1), we get

$$N - \frac{1}{3}K \geqslant 2,$$

whence $2K \leqslant 6N - 12$, $\frac{2K}{N} < 6$. The latter means that there is a face having less than 6 sides. Indeed, let us number the faces and denote by $n_1, n_2, \ldots, n_N$ the number of sides in each face. Then

$$\frac{n_1 + n_2 + \ldots + n_N}{N} = \frac{2K}{N} < 6.$$

327. If each face has more than three sides and from each vertex there emanate more than three edges, then (the same notation as in Problem 324)

$$K \geqslant 2M, \quad K \geqslant 2N$$

and $N - K + M \leqslant 0$, which is impossible.

328. If all the faces are triangles, then the number of edges is multiple of 3. If there is at least one face with the number of sides exceeding three, then the number of edges is not less than eight. An n-gon pyramid has $2n$ edges ($n \geqslant 3$); $(2n + 3)$ edges ($n \geqslant 3$) will be found in the polyhedron which will be obtained if an n-gon pyramid is cut by a triangular plane passing sufficiently close to one of the vertices of the base.

329. If the given polyhedron has n faces, then each face can have from three to $(n - 1)$ sides. Hence it follows that there are two faces with the same number of sides.

330. Consider the so-called d-neighbourhood of our polyhedron, that is, the set of points each of which is found at a distance not greater than d from at least one point of the polyhedron. The surface of the obtained solid

consists of plane parts equal to the corresponding faces of the polyhedron, cylindrical parts corresponding to the edges of the polyhedron (here, if l_i is the length of some edge and α_i is the dihedral angle at this edge, then the surface area of the part of the corresponding cylinder is equal to $(\pi - \alpha_i)\, l_i d$), and spherical parts corresponding to the vertices of the polyhedron the total area of which is equal to the surface area of the sphere of radius d. On the other hand, the surface area of the d-neighbourhood of the polyhedron is less than the surface area of the sphere of radius $d+1$, that is,

$$S + d\sum (\pi - \alpha_i)\, l_i + 4\pi d^2 < 4\pi (d+1)^2.$$

And since $\alpha_i \leqslant \frac{2\pi}{3}$, we get

$$\sum l_i < 24,$$

which was required to be proved.

331. In Fig. 66, O denotes the centre of the sphere, A and B are the points of intersection of the edge of the

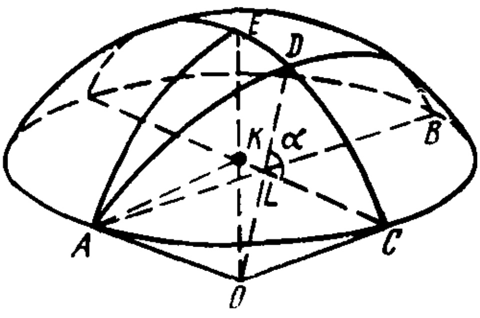


Fig. 66

dihedral angle with the surface of the sphere, D and C are the midpoints of the arcs $\overset{\frown}{ADB}$ and $\overset{\frown}{ACB}$, respectively, the plane ADB passes through O, and E is the vertex of the spherical segment cut off by the plane ACB. The area of the curvilinear triangle ADC amounts to half the desired area. On the other hand $\left(\text{assuming } \alpha \leqslant \frac{\pi}{2}\right)$,

$$S_{ADC} = S_{AEC} - S_{AED}. \qquad (1)$$

Find S_{AEC}. If φ is the angle between the planes AEO and OEC, $|EK|=h$, then obviously, $S_{AEC}=\frac{\varphi}{2\pi}2\pi Rh=\varphi Rh$; h and φ are readily found:

$$h=|EK|=R-|OK|=R-a\sin\alpha,$$

$$\sin\varphi=\sin\widehat{AKL}=\frac{|AL|}{|AK|}=\frac{\sqrt{R^2-a^2}}{\sqrt{R^2-a^2\sin^2\alpha}},$$

$$\varphi=\arcsin\frac{\sqrt{R^2-a^2}}{\sqrt{R^2-a^2\sin^2\alpha}}.$$

Thus,

$$S_{AEC}=R(R-a\sin\alpha)\arcsin\frac{\sqrt{R^2-a^2}}{\sqrt{R^2-a^2\sin^2\alpha}}. \qquad (2)$$

Now find S_{AED}. As is known (see Problem 323),

$$S_{AED}=R^2(\varphi+\psi+\gamma-\pi),$$

where φ, ψ, and γ are the dihedral angles of the trihedral angle with vertex at O and edges OE, OA, and OD. The angle φ is already found.

To determine the angle ψ (the angle at the edge OA), take advantage of the first theorem of cosines (Problem 166) applied to the trihedral angle with vertex A for which

$$\widehat{KAL}=\frac{\pi}{2}-\varphi,\ \sin\widehat{KAO}=\frac{a\sin\alpha}{R},\ \sin\widehat{LAO}=\frac{a}{R}.$$

Consequently,

$$\cos\psi=\frac{\frac{\sqrt{R^2-a^2}}{\sqrt{R^2-a^2\sin^2\alpha}}-\sqrt{1-\frac{a^2\sin^2\alpha}{R^2}}\sqrt{1-\frac{a^2}{R^2}}}{\frac{a\sin\alpha}{R}\cdot\frac{a}{R}}$$

$$=\frac{\sqrt{R^2-a^2}}{\sqrt{R^2-a^2\sin^2\alpha}}\sin\alpha.$$

It is obvious that $\gamma = \pi/2$. Consequently,

$$S_{AED} = R^2 \left[\arcsin \frac{\sqrt{R^2 - a^2}}{\sqrt{R^2 - a^2 \sin^2 \alpha}} + \arccos \frac{\sqrt{R^2 - a^2 \sin \alpha}}{\sqrt{R^2 - a^2 \sin^2 \alpha}} - \frac{\pi}{2} \right]. \qquad (3)$$

Substituting (2) and (3) into (1) and simplifying, we get the answer.

Answer:

$$2R^2 \arccos \frac{R \cos \alpha}{\sqrt{R^2 - a^2 \sin^2 \alpha}} - 2Ra \sin \alpha \arccos \frac{a \cos \alpha}{\sqrt{R^2 - a^2 \sin^2 \alpha}}.$$

332. Consider the regular octahedron with edge $2R$. The ball touching all of its edges has the radius R. The surface of the ball is separated by the surface of the octahedron into eight spherical segments and six curvilinear quadrilaterals equal to the smaller of the two desired.

Answer: $\frac{2\pi R^2}{3}\left(4\sqrt{\frac{2}{3}} - 3\right)$,

$\pi R^2 \left(\frac{16}{3}\sqrt{\frac{2}{3}} - 2\right)$.

333. Twelve lunes with total area $\frac{\pi a^2 (2 - \sqrt{3})}{4}$ and six curvilinear quadrilaterals whose total area is $\frac{\pi a^2 (\sqrt{3} - 1)}{2}$.

334. Suppose that a ball can be inscribed in the given polyhedron. Join the point of tangency of the ball with some face to all the vertices of this face. Each face will be separated into triangles. Triangles situated in neighbouring faces and having a common edge are congruent. Consequently, to each "black" triangle there corresponds a congruent "white" triangle. The sum of the angles of the triangle at each point of tangency is equal to 2π. The

sum of these angles over all faces is equal to $2\pi n$, where n is the number of faces. Of this sum more than half is the share of "black" triangles (by the hypothesis), and the sum of the corresponding angles for "white" triangles, as it was proved, is not less. There is a contradiction.

335. Prove that there can be not more than six balls. Suppose that there are seven balls. Join the centres of all the seven balls to the centre of the given ball and denote by $O_1, O_2, \ldots, O_7$ the points of intersection of these line segments with the surface of the given ball. For each point O_i consider on the sphere the set of points for which the distance (over the surface of the sphere) to the point O_i is not greater than the distance to any other point O_k, $k \neq i$. The sphere will be separated into seven spherical polygons. Each polygon is the intersection of six hemispheres containing the point O_i whose boundary is the great circle along which the plane passing through the midpoint O_iO_k and perpendicular to it cuts the sphere.

Each of the formed polygons contains a circle whose spherical radius is seen from the centre of the original sphere at an angle α, $\sin \alpha = 0.7$.

Denote by K and N, respectively, the number of sides and vertices of the separation thus obtained. (Each side is a common side of two adjacent polygons and is counted only once. The same is valid for the vertices.) It is easily seen that for such separation Euler's formula holds true (see Problem 324). In our case this will yield $K = N + 5$. On the other hand, $K \geqslant \frac{3}{2}N$, since from each vertex there emanate at least three sides, and each side is counted twice.

Now, it is easy to obtain that $K \leqslant 15$, $N \leqslant 10$. In Problem 325, we have proved that among all spherical n-gons containing the given circle a regular n-gon has the smallest area. Besides, it is possible to show that the sum of areas of regular n- and $(n+2)$-gons is greater than the doubled area of a regular n-gon. (The polygons circumscribed about one circle are considered.) It is also obvious that the area of a regular circumscribed n-gon is decreased with an increase in n. Hence it follows that the sum of areas of the seven obtained polygons cannot be less than the sum of areas of five regular quadrilaterals and two regular pentagons circumscribed about the circle with the spherical radius to which there corresponds the central angle $\alpha = \arcsin 0.7$. The area of the corresponding regular pentagon will be

$$s_5 = 9\left[10 \arccos\left(\cos\alpha \sin\frac{\pi}{5}\right) - 3\pi\right],$$

the area of the regular quadrilateral

$$s_4 = 9\left[8 \arccos\left(\frac{\sqrt{2}}{2}\cos\alpha\right) - 2\pi\right].$$

We can readily prove that $2s_5 + 5s_4 > 36\pi$. Thus, seven balls with radius 7 cannot simultaneously touch the ball with radius 3 without intersecting one another. At the same time we can easily show that it is possible in the case of six balls.

336. Consider the cube $ABCDA_1B_1C_1D_1$. On the edges A_1B and A_1D take points K and L such that $|A_1K| = |CM|$, $|A_1L| = |CN|$. Let P and Q denote the points of intersection of the lines AK and BA_1, AL and DA_1, respectively.

As is easily seen, the sides of the triangle A_1PQ are equal to the corresponding line segments of the diagonal BD. And since the triangle BA_1D is regular, our statement has been proved.

337. If the point P did not lie in the plane of the triangle ABC, the statement of the problem would be obvious, since in that case the points P, A_2, B_2, and C_2 would belong to the section of the surface of the sphere circumscribed about the tetrahedron $ABCP$ by the plane passing through P and l. The statement of our problem can now be obtained with the aid of the passage to the limit.

338. Let $ABCDEF$ denote the plane hexagon circumscribed about the circle. Take an arbitrary space hexagon $A_1B_1C_1D_1E_1F_1$ (Fig. 67), different from $ABCDEF$, whose projection on our plane is the hexagon $ABCDEF$ and whose corresponding sides pass through the points of contact of the hexagon $ABCDEF$ and the circle. To prove the existence of such hexagon $A_1B_1C_1D_1E_1F_1$, it suffices to take one vertex, say A_1, arbitrarily on the perpendicular to the plane erected at the point A, then the remain-

ing vertices will be determined identically. Indeed, let a, b, c, d, e, and f be the lengths of the tangents to the circle drawn through the respective points A, B, C, D, E, F, and h the distance from A to the plane. Then B_1 lies on the other side of the plane as compared with A

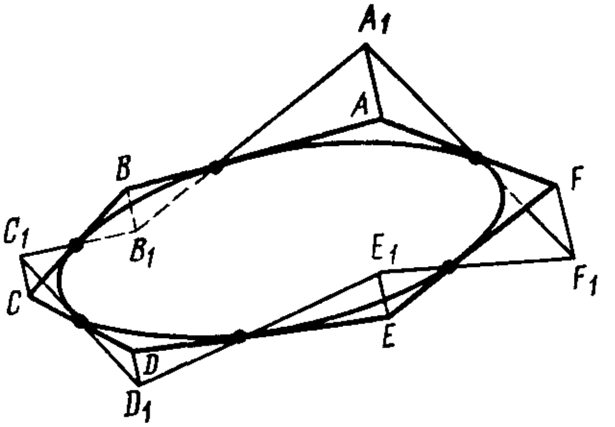


Fig. 67

at a distance of $\frac{hb}{a}$, C_1 on the same side as A_1 at a distance of $\frac{hb}{a} \cdot \frac{c}{b} = \frac{hc}{a}$ from the plane, and so on. Finally, we find that F_1 lies on the other side of the plane as compared with A_1 at a distance of $\frac{hf}{a}$ and, hence, A_1 and F_1 lie on the straight line passing through the point of tangency of AF with the circle.

Any two opposite sides of the hexagon $A_1B_1C_1D_1E_1F_1$ lie in one and the same plane. This follows from the fact that all the angles formed by the sides of the hexagon with the given plane are congruent. Consequently, any two diagonals connecting the opposite vertices of the hexagon $A_1B_1C_1D_1E_1F_1$ intersect, and, hence, all the three diagonals of this hexagon (they do not lie in one plane) intersect at one point. Since the hexagon $ABCDEF$ is the projection of the hexagon $A_1B_1C_1D_1E_1F_1$, the theorem has been proved.

339. The plane configuration indicated in the problem can be regarded as three-dimensional projection: a tri-

hedral angle cut by two planes, for which our statement is obvious.

340. This problem represents one of the possible three-dimensional analogues of Desargues' theorem (see Problem 339). For its solution, it is convenient to go out to a four-dimensional space.

Let us first consider some properties of this space.

The simplest figures of the four-dimensional space are: a point, a straight line, a plane, and a three-dimensional variety which will be called the *hyperplane*. The first three figures are our old friends from the three-dimensional space. Of course, some statements concerning these figures must be refined. For instance, the following axiom of the three-dimensional space: if two distinct planes have a common point, then they intersect along a straight line, must be replaced by the axiom: if two distinct planes belonging to one hyperplane have a common point, then they intersect along a straight line. The introduction of a new geometric image, a hyperplane, prompts the necessity to introduce a group of relevant axioms, just as the passage from plane geometry to solid geometry requires a group of new axioms (refresh them, please) expressing the basic properties of planes in space. This group consists of the following three axioms:

1. Whatever a hyperplane is, there are points belonging to it and points not belonging to it.

2. If two distinct hyperplanes have a common point, then they intersect over a plane, that is, there is a plane belonging to each of the hyperplanes.

3. If a straight line not belonging to a plane has a common point with this plane, then there is a unique hyperplane containing this line and this plane.

From these axioms it follows directly that four points not belonging to one plane determine a hyperplane; exactly in the same way, three straight lines not belonging to one plane, but having a common point, or two distinct planes having a common straight line determine a hyperplane. We are not going to prove these statements, try to do it independently.

For our further reasoning we need the following fact existing in the four-dimensional space: three distinct hyperplanes having a common point also have a common straight line. Indeed, by Axiom 2, any two of three hyperplanes have a common plane. Let us take two planes over which one of the three hyperplanes intersects with two others. These two planes belonging to one hyperplane have a common point and, hence, intersect along a straight line or coincide.

Let us now pass to the proof of our statement. If the three planes under consideration were arranged in a four-dimensional space, then the statement would be obvious. Indeed, every trihedral angle determines a hyperplane. Two hyperplanes intersect over a plane. This plane does not belong to a third hyperplane (by the hypothesis, these hyperplanes intersect one of the given planes along three straight lines not passing through one point) and, consequently, intersects with them along a straight line. Any three corresponding faces of trihedral angles lie in one hyperplane determined by two planes on whicb the corresponding edges lie, and therefore each triple of the corresponding faces has a common point. These three points belong to the three hyperplanes determined by the trihedral angles, and, as it was proved, lie on one straight line. Now, to complete the proof, it is sufficient to "see" in the given hypothesis the projection of the corresponding four-dimensional configuration of planes and trihedral angles.

TO THE READER

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

Our address is:
Mir Publishers
2 Pervy Rizhsky Pereulok
I-110, GSP, Moscow, 129820
USSR

Printed in the Union of Soviet Socialist Republics